PART B

Unit - VII

Laplace Transforms - 1

7.1 Introduction

Many physical problems when analysed assumes the form of an ordinary differential equation subjected to a set of initial conditions or boundary conditions. Such problems are referred to as initial value problems and boundary value problems respectively.

Laplace Transforms (L.T) serves as a very useful tool in solving these problems without actually finding the general solution of the differential equation by various known methods.

The application of Laplace transforms is highly significant in engineering problems associated with electric circuits.

7.2 Definition

If $f(t)$ is a real valued function defined for all $t \geq 0$ then the ***Laplace transform*** of $f(t)$ denoted by $L[f(t)]$ is defined by

$$L[f(t)] = \int_{t=0}^{\infty} e^{-st} f(t)\, dt$$

provided the integral exists. On integration of the indefinite integral we will be having a function of s and t. When this is evaluated between the limits $t=0$ and $t=\infty$ we will be left with a function of s only and we shall denote it by $\overline{f}(s)$ where s is a parameter, real or complex. Thus

$$L[f(t)] = \overline{f}(s).$$

Equivalently we can express this in the form

$$L^{-1}[\overline{f}(s)] = f(t)$$

and is called the ***inverse Laplace transform.***

Note : *Since linearity property holds good for an integral we can obviously infer that* $L[c_1 f_1(t) \pm c_2 f_2(t)] = c_1 L[f_1(t)] \pm c_2 L[f_2(t)]$ *where* c_1 *&* c_2 *are constants.*

7.3 Laplace transform of discontinuous functions

Working procedure for problems

- Here $f(t)$ is composed of different functions in different subintervals of $(0, \infty)$
- Suppose $f(t) = \begin{cases} f_1(t), & 0<t<a \\ f_2(t), & a<t<b \\ f_3(t), & t>b \end{cases}$

  then we consider the basic definition of $L[f(t)]$. That is

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

- By taking into account the intervals involved in the given $f(t)$ we express the integral in the equivalent form by using a property of definite integrals.

$$\therefore \quad L[f(t)] = \int_0^a e^{-st} f(t)\, dt + \int_a^b e^{-st} f(t)\, dt + \int_b^\infty e^{-st} f(t)\, dt$$

- We substitute the relevant $f(t)$ as in the data.

$$\text{Now} \quad L[f(t)] = \int_0^a e^{-st} f_1(t)\, dt + \int_a^b e^{-st} f_2(t)\, dt + \int_b^\infty e^{-st} f_3(t)\, dt$$

- We evaluate the integrals to obtain $L[f(t)]$ as a function of s.

Note : *If $F(t)$ is a polynomial in t then we have to write $\int e^{-st} F(t)\, dt$ as $\int F(t)\, e^{-st}\, dt$ and Bernoulli's generalized rule of integration by parts as follows will be convenient to complete the process of integration.*

$$\int u\, v\, dt = u \int v\, dt - u' \iint v\, dt\, dt + u'' \iiint v\, dt\, dt\, dt - \cdots$$

WORKED PROBLEMS

1. *Find $L[f(t)]$ where $f(t) = \begin{cases} t, & 0<t<4 \\ 5, & t>4 \end{cases}$*

$$>> \quad L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^4 e^{-st} f(t)\, dt + \int_4^\infty e^{-st} f(t)\, dt$$

Using the relevant $f(t)$ in the integrals we have,

$$L[f(t)] = \int_0^4 e^{-st}\cdot t\,dt + \int_4^\infty e^{-st}\cdot 5\,dt$$

$$= \int_0^4 t\,e^{-st}\,dt + 5\int_4^\infty e^{-st}\,dt$$

Using Bernoulli's rule for the first term in R.H.S we have,

$$L[f(t)] = \left[t\cdot\frac{e^{-st}}{-s}\right]_{t=0}^{4} - \left[1\cdot\frac{e^{-st}}{s^2}\right]_{t=0}^{4} + 5\left[\frac{e^{-st}}{-s}\right]_4^\infty$$

$$= \frac{-1}{s}(4e^{-4s}-0) - \frac{1}{s^2}(e^{-4s}-1) - \frac{5}{s}(0-e^{-4s})$$

Thus $\quad L[f(t)] = \frac{1}{s}e^{-4s} + \frac{1}{s^2}(1-e^{-4s})$

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2. *Find* $L[f(t)]$ *if* $f(t) = \begin{cases}\sin 2t, & 0<t<\pi\\ 0, & t>\pi\end{cases}$

\>> $\quad L[f(t)] = \int_0^\infty e^{-st}f(t)\,dt = \int_0^\pi e^{-st}f(t)\,dt + \int_\pi^\infty e^{-st}f(t)\,dt$

ie., $\quad = \int_0^\pi e^{-st}\sin 2t\,dt + \int_\pi^\infty e^{-st}\cdot 0\,dt$

Using $\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$ we have,

$$L[f(t)] = \left[\frac{e^{-st}}{(-s)^2+4}(-s\sin 2t - 2\cos 2t)\right]_{t=0}^{\pi} + 0$$

$$= \frac{-1}{s^2+4}\left[e^{-st}(s\sin 2t + 2\cos 2t)\right]_{t=0}^{\pi}$$

$$= \frac{-1}{s^2+4}[e^{-s\pi}\cdot 2 - 2]$$

$\because \quad \cos 2\pi = 1 = \cos 0,\ \sin 2\pi = 0 = \sin 0$

Thus $\quad L[f(t)] = \frac{2}{s^2+4}(1-e^{-\pi s})$

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3. *Find* $L[f(t)]$ *if* $f(t) = \begin{cases} e^t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$

>> $$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st} f(t)\,dt + \int_1^\infty e^{-st} f(t)\,dt$$

ie., $$= \int_0^1 e^{-st}\cdot e^t\,dt + 0 = \int_0^1 e^{-(s-1)t}\,dt$$

ie., $$= \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^1 = \frac{-1}{s-1}\left[e^{-(s-1)} - 1\right]$$

Thus $$\mathbf{L[f(t)] = \frac{1}{s-1}\left[1 - e^{-(s-1)}\right]}$$

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4. *Find* $L[f(t)]$ *if* $f(t) = \begin{cases} 0, & 0 < t < 1 \\ t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$

>> $$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} f(t)\,dt + \int_1^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

On substituting for $f(t)$ we have only one integral in the R.H.S.

$\therefore$ $$L[f(t)] = \int_1^2 e^{-st}\cdot t\,dt = \int_1^2 t\,e^{-st}\,dt$$

Applying Bernoulli's rule,

$$L[f(t)] = \left[t\left(\frac{e^{-st}}{-s}\right) - 1\left(\frac{e^{-st}}{s^2}\right)\right]_1^2$$

$$= \frac{-1}{s}(2e^{-2s} - e^{-s}) - \frac{1}{s^2}(e^{-2s} - e^{-s})$$

Thus $$\mathbf{L[f(t)] = \frac{1}{s}(e^{-s} - 2e^{-2s}) + \frac{1}{s^2}(e^{-s} - e^{-2s})}$$

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5. *If* $f(t) = \begin{cases} \cos(t - 2\pi/3), & t > 2\pi/3 \\ 0, & 0 < t < 2\pi/3 \end{cases}$ *find* $L[f(t)]$

\>> $$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt = \int_0^{2\pi/3} e^{-st} f(t)\,dt + \int_{2\pi/3}^{\infty} e^{-st} f(t)\,dt$$

$$L[f(t)] = 0 + \int_{2\pi/3}^{\infty} e^{-st}\cos(t - 2\pi/3)\,dt$$

$$= \left\{ \frac{e^{-st}}{s^2+1}\left[-s\cos(t-2\pi/3) + \sin(t-2\pi/3)\right] \right\}_{2\pi/3}^{\infty}$$

$$L[f(t)] = \frac{1}{s^2+1}\left\{0 - e^{-2\pi s/3}(-s\cos 0 + \sin 0)\right\}$$

Thus $$L[f(t)] = \frac{s\,e^{-2\pi s/3}}{s^2+1}$$

7.4 Laplace transform of some standard functions

1. *$L(a)$ where 'a' is a constant.*

$$L(a) = \int_0^{\infty} e^{-st}\cdot a\,dt = a\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{-a}{s}(0-1) = \frac{a}{s}$$

Thus $L(a) = \dfrac{a}{s}$ where $s > 0$. If $a = 1$ then $L(1) = \dfrac{1}{s}$

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2. $L(e^{at})$

$$L(e^{at}) = \int_0^{\infty} e^{-st}\cdot e^{at}\,dt = \int_0^{\infty} e^{-(s-a)t}\,dt$$

$$L(e^{at}) = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty} = 0 - \frac{1}{-(s-a)} = \frac{1}{s-a}$$

Thus $L(e^{at}) = \dfrac{1}{s-a}$ where $s > a$

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3. $L(\cosh at)$

$$L(\cosh at) = L\left(\frac{e^{at}+e^{-at}}{2}\right) = \frac{1}{2}\left\{L(e^{at})+L(e^{-at})\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a}+\frac{1}{s+a}\right\} \text{ since } L(e^{at}) = \frac{1}{s-a}$$

$$= \frac{1}{2}\left\{\frac{(s+a)+(s-a)}{(s-a)(s+a)}\right\} = \frac{1}{2}\cdot\frac{2s}{s^2-a^2} = \frac{s}{s^2-a^2}$$

Thus $\quad \mathbf{L(\cosh at) = \frac{s}{s^2-a^2}}$ where $s > a$

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4. $L(\sinh at)$

$$L(\sinh at) = L\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{1}{2}\left\{L(e^{at})-L(e^{-at})\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a}-\frac{1}{s+a}\right\} = \frac{1}{2}\left\{\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right\} = \frac{a}{s^2-a^2}$$

Thus $\quad \mathbf{L(\sinh at) = \frac{a}{s^2-a^2}}$ where $s > a$

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5. $L(\cos at)$

$$L(\cos at) = \int_0^\infty e^{-st}\cos at\,dt$$

Using $\int e^{at}\cos bt\,dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)$ we have,

$$L(\cos at) = \left[\frac{e^{-st}}{(-s)^2+a^2}(-s\cos at + a\sin at)\right]_{t=0}^{\infty}$$

$$= \frac{1}{s^2+a^2}\left[e^{-st}(-s\cos at + a\sin at)\right]_{t=0}^{\infty}$$

$$= \frac{1}{s^2+a^2}\left[0 - e^0(-s\cos 0 + a\sin 0)\right] = \frac{s}{s^2+a^2}$$

Thus $\quad \mathbf{L(\cos at) = \frac{s}{s^2+a^2}}$ where $s > 0$

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6. $L(\sin at)$

$$L(\sin at) = \int_0^\infty e^{-st} \sin at \, dt$$

Using $\int e^{at} \sin bt \, dt = \dfrac{e^{at}}{a^2+b^2}(a \sin bt - b \cos bt)$ we have,

$$L(\sin at) = \left[\frac{e^{-st}}{(-s)^2+a^2}(-s \sin at - a \cos at) \right]_{t=0}^{\infty}$$

$$= \frac{-1}{s^2+a^2}\left[e^{-st}(s \sin at + a \cos at) \right]_0^\infty = \frac{-1}{s^2+a^2}(0-a) = \frac{a}{s^2+a^2}$$

Thus $\quad \mathbf{L(\sin at) = \dfrac{a}{s^2+a^2}}$ where $s > 0$

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7. $L(t^n)$

$$L(t^n) = \int_0^\infty e^{-st} t^n \, dt$$

Put $st = x \;\therefore\; dt = dx/s$ and x varies from 0 to ∞

Now $\quad L(t^n) = \displaystyle\int_{x=0}^{\infty} e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n \, dx = \frac{\Gamma(n+1)}{s^{n+1}}$

Thus $\quad \mathbf{L(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}}$ where n is a constant.

Remarks

1. *We know that $\Gamma(n+1)$ exists if n is a positive real number or n is a non negative integer. Hence the expression for $L(t^n)$ is valid for n belonging to these categories of n.*

2. *We know that $\Gamma(n+1) = n!$ if n is a positive integer.*

$\therefore \quad L(t^n) = \dfrac{n!}{s^{n+1}}$ *if n is a positive integer.*

We shall establish this result without the involvement of gamma functions.

8. $L(t^n)$ *where n is a positive integer.*

$$L(t^n) = \int_0^\infty e^{-st} t^n \, dt = \int_0^\infty t^n e^{-st} \, dt$$

Integrating by parts we have,

$$L(t^n) = \left[t^n \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} (n t^{n-1}) \, dt = 0 + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} \, dt$$

$\therefore \quad L(t^n) = \frac{n}{s} L(t^{n-1})$

Similarly $L(t^{n-1}) = \frac{n-1}{s} L(t^{n-2})$, $L(t^{n-2}) = \frac{n-2}{s} L(t^{n-3})$ etc.

Also $L(t^2) = \frac{2}{s} L(t^1)$, $L(t^1) = \frac{1}{s} L(t^0)$

Using all these results by back substitution we have,

$$L(t^n) = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} L(t^0)$$

$$= \frac{n!}{s^n} L(1) = \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$$

Thus $\boldsymbol{L(t^n) = \frac{n!}{s^{n+1}}}$ where n is a positive integer.

Table of Laplace transforms

	$f(t)$	$L[f(t)] = \overline{f}(s)$		$f(t)$	$L[f(t)] = \overline{f}(s)$
1.	a	$\frac{a}{s}$	5.	$\sinh at$	$\frac{a}{s^2 - a^2}$
2.	e^{at}	$\frac{1}{s-a}$	6.	$\sin at$	$\frac{a}{s^2 + a^2}$
3.	$\cosh at$	$\frac{s}{s^2 - a^2}$	**7.**	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$
4.	$\cos at$	$\frac{s}{s^2 + a^2}$	**8.**	t^n, $n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$

Observe the following illustrations based on the table of Laplace transforms.

1. $L(4) = \dfrac{4}{s}$

2. $L(e^{2t}) = \dfrac{1}{s-2}$

3. $L(e^{-t}) = \dfrac{1}{s+1}$

4. $L(3e^{4t}) = \dfrac{3}{s-4}$

5. $L(2\cosh 2t) = \dfrac{2s}{s^2-4}$

6. $L(3\sinh 2t) = \dfrac{6}{s^2-4}$

7. $L(\cos t) = \dfrac{s}{s^2+1}$

8. $L(3\cos 4t) = \dfrac{3s}{s^2+16}$

9. $L(t^5) = \dfrac{5!}{s^6} = \dfrac{120}{s^6}$

10. $L(4t^3) = 4 \cdot \dfrac{3!}{s^4} = \dfrac{24}{s^4}$

11. $L(\sqrt{t}) = L(t^{1/2}) = \dfrac{\Gamma(3/2)}{s^{3/2}}$

12. $L(\sin 2t) = \dfrac{2}{s^2+4}$

ie., $L(\sqrt{t}) = \dfrac{1/2 \cdot \Gamma(1/2)}{s^{3/2}} = \dfrac{\sqrt{\pi}}{2\,s^{3/2}}$

WORKED PROBLEMS

Find the Laplace transform of the following functions

6. $\cosh^2 3t$

7. $e^{-2t} \sinh 4t$

8. $\sin 5t \cdot \cos 2t$

9. $\cos t \cdot \cos 2t \cdot \cos 3t$

10. $\sin^2(2t+1)$

11. $(3t+4)^3 + 5^t$

12. $3\sqrt{t} + \dfrac{4}{\sqrt{t}}$

13. $\left\{\sqrt{t} - \dfrac{1}{\sqrt{t}}\right\}^3$

14. $t^{-5/2} + t^{5/2}$

6. Let $f(t) = \cosh^2 3t = \left[\dfrac{e^{3t}+e^{-3t}}{2}\right]^2$

ie., $$f(t) = \frac{1}{4}(e^{6t} + e^{-6t} + 2)$$

Thus $$L[f(t)] = \frac{1}{4}\left[\frac{1}{s-6} + \frac{1}{s+6} + \frac{2}{s}\right]$$

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7. Let $f(t) = e^{-2t} \cdot \sinh 4t = e^{-2t} \cdot \left(\dfrac{e^{4t} - e^{-4t}}{2} \right)$

ie., $$f(t) = \frac{1}{2}(e^{2t} - e^{-6t})$$

Thus $$L[f(t)] = \frac{1}{2}\left(\frac{1}{s-2} - \frac{1}{s+6} \right) = \frac{4}{(s-2)(s+6)}$$

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8. Let $f(t) = \sin 5t \cos 2t = \dfrac{1}{2}\{ \sin(5t+2t) + \sin(5t-2t) \}$

ie., $$f(t) = \frac{1}{2}\{ \sin 7t + \sin 3t \}$$

Thus $$L[f(t)] = \frac{1}{2}\left[\frac{7}{s^2+49} + \frac{3}{s^2+9} \right]$$

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9. Let $f(t) = \cos t \cos 2t \cos 3t$

Now $\cos t \cdot \cos 2t = \dfrac{1}{2}\{ \cos(t+2t) + \cos(t-2t) \} = \dfrac{1}{2}(\cos 3t + \cos t)$

Thus $$\cos t \cos 2t \cos 3t = \frac{1}{2}\{ \cos 3t \cos 3t + \cos 3t \cos t \}$$

ie., $$f(t) = \frac{1}{2}\left\{ \frac{1}{2}(\cos 6t + \cos 0) + \frac{1}{2}(\cos 4t + \cos 2t) \right\}$$

$$f(t) = \frac{1}{4}(\cos 6t + 1 + \cos 4t + \cos 2t)$$

Thus $$L[f(t)] = \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{1}{s} + \frac{s}{s^2+16} + \frac{s}{s^2+4} \right]$$

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10. Let $f(t) = \sin^2(2t+1) = \dfrac{1}{2}\{ 1 - \cos 2(2t+1) \}$

ie., $$f(t) = \frac{1}{2}\{ 1 - \cos(4t+2) \}$$

ie., $$f(t) = \frac{1}{2}\{ 1 - \cos 4t \cdot \cos 2 + \sin 4t \sin 2 \}$$

Thus $$L[f(t)] = \frac{1}{2}\left\{ \frac{1}{s} - \frac{s \cos 2}{s^2+16} + \frac{4 \sin 2}{s^2+16} \right\}$$

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11. Let $f(t) = (3t+4)^3 + 5^t$

ie., $f(t) = (27t^3 + 108t^2 + 144t + 64) + e^{\log 5 \cdot t}$

$$\therefore \quad L[f(t)] = 27 \cdot \frac{3!}{s^4} + 108 \cdot \frac{2!}{s^3} + 144 \cdot \frac{1!}{s^2} + \frac{64}{s} + \frac{1}{s - \log 5}$$

Thus $\mathbf{L[f(t)] = \frac{162}{s^4} + \frac{216}{s^3} + \frac{144}{s^2} + \frac{64}{s} + \frac{1}{s - \log 5}}$

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12. Let $f(t) = 3\sqrt{t} + \frac{4}{\sqrt{t}} = 3t^{1/2} + 4t^{-1/2}$

$$\therefore \quad L[f(t)] = 3\frac{\Gamma(3/2)}{s^{3/2}} + 4\frac{\Gamma(1/2)}{s^{1/2}} = \frac{3\sqrt{\pi}}{2s^{3/2}} + \frac{4\sqrt{\pi}}{\sqrt{s}}$$

Thus $\mathbf{L[f(t)] = \sqrt{\frac{\pi}{s}}\left[\frac{3}{2s} + 4\right]}$

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13. Let $f(t) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3 = t^{3/2} - \frac{1}{t^{3/2}} - 3\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)$

ie., $f(t) = t^{3/2} - t^{-3/2} - 3t^{1/2} + 3t^{-1/2}$

$$L[f(t)] = \frac{\Gamma(5/2)}{s^{5/2}} - \frac{\Gamma(-1/2)}{s^{-1/2}} - \frac{3\Gamma(3/2)}{s^{3/2}} + \frac{3\Gamma(1/2)}{s^{1/2}} \quad \ldots (1)$$

$$\Gamma(1/2) = \sqrt{\pi}, \ \Gamma(3/2) = \frac{\sqrt{\pi}}{2}, \ \Gamma(5/2) = \frac{3\sqrt{\pi}}{4},$$

$$\Gamma(-1/2) = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}$$

Substituting these values in (1) we get,

$$L[f(t)] = \frac{3\sqrt{\pi}}{4s^{5/2}} + 2\sqrt{\pi}\sqrt{s} - \frac{3\sqrt{\pi}}{2s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}}$$

Thus $\mathbf{L[f(t)] = \sqrt{\pi}\left[\frac{3}{4s^2\sqrt{s}} + 2\sqrt{s} - \frac{3}{2s\sqrt{s}} + \frac{3}{\sqrt{s}}\right]}$

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14. Let $f(t) = t^{-5/2} + t^{5/2}$

$$\therefore \quad L[f(t)] = \frac{\Gamma(-3/2)}{s^{-3/2}} + \frac{\Gamma(7/2)}{s^{7/2}} \qquad \ldots (1)$$

But $$\Gamma(-3/2) = \frac{\Gamma(-1/2)}{-3/2} = \frac{-2}{3} \cdot \frac{\Gamma(1/2)}{-1/2} = \frac{4\sqrt{\pi}}{3}$$

$$\Gamma(7/2) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi} = \frac{15\sqrt{\pi}}{8}$$

Substituting these values in (1) we get,

$$L[f(t)] = \frac{4\sqrt{\pi}}{3\,s^{-3/2}} + \frac{15\sqrt{\pi}}{8\,s^{7/2}}$$

Thus $$L[f(t)] = \sqrt{\pi}\left[\frac{4}{3}s^{3/2} + \frac{15}{8}\frac{1}{s^{7/2}}\right]$$

7.5 Properties of Laplace transforms

7.51 If $L[f(t)] = \bar{f}(s)$ then $L[e^{at} f(t)] = \bar{f}(s-a)$

Proof : We have by the definition,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s) \qquad \ldots (1)$$

$$\therefore \quad L[e^{at} f(t)] = \int_0^\infty e^{-st}[e^{at} f(t)]\,dt = \int_0^\infty e^{-st+at} f(t)\,dt$$

ie., $$L[e^{at} f(t)] = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

Comparing this integral with the integral in (1) being denoted by $\bar{f}(s)$ we observe that $(s-a)$ has replaced s.

Thus $L[e^{at} f(t)] = \bar{f}(s-a)$

Note : $L[e^{-at} f(t)] = \bar{f}(s+a)$

Remark : *This property is known as the shifting property.*

7.52 **If $L[f(t)] = \bar{f}(s)$, then $L[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n}[\bar{f}(s)]$ where n is a positive integer.**

Proof : We establish the result by the principle of mathematical induction.

We have $\quad \bar{f}(s) = \displaystyle\int_0^\infty e^{-st} f(t)\, dt$

Differentiating w.r.t.s on both sides we have,

$$\frac{d}{ds}[\bar{f}(s)] = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}] f(t)\, dt.$$

In the R.H.S, we shall apply Leibnitz rule for differentiation under the integral sign.

$$\therefore \quad \frac{d}{ds}[\bar{f}(s)] = \int_0^\infty e^{-st}(-t) f(t)\, dt$$

or $$(-1)\frac{d}{ds}[\bar{f}(s)] = \int_0^\infty e^{-st}[t f(t)]\, dt = L[t f(t)] \qquad \ldots (1)$$

This verifies the result for $n = 1$

Let us assume the result to be true for $n = k$

ie., $$(-1)^k \frac{d^k}{ds^k}\left[\bar{f}(s)\right] = L[t^k f(t)] \qquad \ldots (2)$$

or $$(-1)^k \frac{d^k}{ds^k}\left[\bar{f}(s)\right] = \int_0^\infty e^{-st}[t^k f(t)]\, dt$$

Differentiating w.r.t.s again we get,

$$(-1)^k \frac{d^{k+1}}{ds^{k+1}}\left[\bar{f}(s)\right] = \int_0^\infty \frac{\partial}{\partial s}(e^{-st})[t^k f(t)]\, dt$$

ie., $$(-1)^k \frac{d^{k+1}}{ds^{k+1}}\left[\bar{f}(s)\right] = \int_0^\infty e^{-st}(-t)[t^k f(t)]\, dt$$

Multiplying by (-1) we get,

$$(-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}}\left[\bar{f}(s)\right] = \int_0^\infty e^{-st}[t^{k+1} f(t)]\, dt$$

ie., $\quad (-1)^{k+1}\dfrac{d^{k+1}}{ds^{k+1}}\left[\bar{f}(s)\right] = L[\,t^{k+1}f(t)\,]$ ...(3)

Comparing (2) and (3) we conclude that the result is true for $n = k+1$. Hence by the principle of mathematical induction the result is true for all positive integral values of n.

Thus $\quad L[\,t^n f(t)\,] = (-1)^n \dfrac{d^n}{ds^n}[\bar{f}(s)] = (-1)^n\ \bar{f}^{(n)}(s)$

Remark : *This property is called the* ***derivative of the transform*** *property.*

7.53 **If** $L[f(t)] = \bar{f}(s)$ **then** $L\left[\dfrac{f(t)}{t}\right] = \displaystyle\int_s^\infty \bar{f}(s)\,ds$

Proof : We have $\bar{f}(s) = \displaystyle\int_0^\infty e^{-st}f(t)\,dt$

$$\therefore \quad \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty\left[\int_0^\infty e^{-st}f(t)\,dt\right]ds$$

ie., $$= \int_0^\infty\int_s^\infty e^{-st}f(t)\,ds\,dt, \text{ on changing the order of integration.}$$

$$= \int_0^\infty\left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)\,dt = \int_0^\infty\left[0-\frac{e^{-st}}{-t}\right]f(t)\,dt$$

ie., $$\int_s^\infty \bar{f}(s)\,ds = \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right]dt = L\left[\frac{f(t)}{t}\right]$$

Thus $$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$$

7.54 **If** $L[f(t)] = \bar{f}(s)$ **then** $L\left[\displaystyle\int_0^t f(t)\,dt\right] = \dfrac{\bar{f}(s)}{s}$

Proof : Let $F(t) = \displaystyle\int_0^t f(t)\,dt$ and hence $F'(t) = f(t)$ and $F(0) = 0$

Now $L[F(t)] = \int_0^{\infty} F(t)\, e^{-st}\, dt$ by the definition.

Integrating by parts we get,

$$L[F(t)] = \left[F(t)\frac{e^{-st}}{-s}\right]_{t=0}^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} F'(t)\, dt$$

$$= (0-0) + \frac{1}{s}\int_0^{\infty} e^{-st} f(t)\, dt = \frac{1}{s}\, \bar{f}(s)$$

Thus $\mathbf{L}\left[\int_0^t f(t)\, dt\right] = \frac{\bar{f}(s)}{s}$

Properties at a glance

If $L[f(t)] = \bar{f}(s)$ then we have,

1. $L[e^{at} f(t)] = \bar{f}(s-a)$
2. $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$

In particular $L[t f(t)] = -\frac{d}{ds}\left[\bar{f}(s)\right]$, $L[t^2 f(t)] = \frac{d^2}{ds^2}[\bar{f}(s)]$

3. $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(s)\, ds$
4. $L\left[\int_0^t f(t)\, dt\right] = \frac{\bar{f}(s)}{s}$

WORKED PROBLEMS

Find the Laplace transform of the following functions

15. $e^{-2t}(2\cos 5t - \sin 5t)$
16. $e^{-t}\cos^2 3t$
17. $e^{-4t}\, t^{-5/2}$
18. $(1 + 3t\, e^{2t})^2$
19. $e^{3t}\sin 5t \sin 3t$
20. $\sinh at \sin at$
21. $\cosh t \sin^3 2t$

15. Let $f(t) = 2\cos 5t - \sin 5t$

$$\therefore \quad L[f(t)] = 2 \cdot \frac{s}{s^2+5^2} - \frac{5}{s^2+5^2} = \frac{2s-5}{s^2+25}$$

Now $\quad L[e^{-2t} f(t)] = \left\{ \frac{2s-5}{s^2+25} \right\}_{s \to s+2} = \frac{2(s+2)-5}{(s+2)^2+25}$

Thus $\quad \mathbf{L[e^{-2t}(2\cos 5t - \sin 5t)] = \frac{2s-1}{s^2+4s+29}}$

---

16. Let $f(t) = \cos^2 3t = \frac{1+\cos 6t}{2}$

$$\therefore \quad L[f(t)] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

Now $\quad L[e^{-t}\cos^2 3t] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]_{s \to (s+1)}$

Thus $\quad \mathbf{L[e^{-t}\cos^2 3t] = \frac{1}{2}\left[\frac{1}{s+1} + \frac{(s+1)}{(s+1)^2+36}\right]}$

---

17. Let $f(t) = t^{-5/2}$

$$\therefore \quad L[f(t)] = \frac{4\sqrt{\pi}}{3} s^{3/2} \quad \textit{(Refer Problem-14)}$$

Now $\quad L[e^{-4t} t^{-5/2}] = \frac{4\sqrt{\pi}}{3}[s^{3/2}]_{s \to s+4}$

Thus $\quad \mathbf{L[e^{-4t} t^{-5/2}] = \frac{4\sqrt{\pi}}{3}(s+4)^{3/2}}$

---

18. Let $f(t) = (1+3t e^{2t})^2 = 1 + 6t e^{2t} + 9t^2 e^{4t}$

$$\therefore \quad L[f(t)] = L(1) + 6L(e^{2t} \cdot t) + 9L(e^{4t} \cdot t^2)$$

$$= \frac{1}{s} + 6\{L(t)\}_{s \to (s-2)} + 9\{L(t^2)\}_{s \to (s-4)}$$

But $\quad L(t) = \frac{1}{s^2}$ and $L(t^2) = \frac{2}{s^3}$

Thus $\quad \mathbf{L[(1+3te^{2t})^2] = \frac{1}{s} + \frac{6}{(s-2)^2} + \frac{18}{(s-4)^3}}$

---

19. Let $f(t) = \sin 5t \sin 3t = \frac{1}{2}\{\cos(5t-3t) - \cos(5t+3t)\}$

ie., $f(t) = \frac{1}{2}(\cos 2t - \cos 8t)$

$\therefore \quad L[f(t)] = \frac{1}{2}\left(\frac{s}{s^2+4} - \frac{s}{s^2+64}\right) = \frac{30s}{(s^2+4)(s^2+64)}$

Hence $L[e^{3t} f(t)] = \left\{\frac{30s}{(s^2+4)(s^2+64)}\right\}_{s \to (s-3)}$

ie., $L[e^{3t}\sin 5t \sin 3t] = \frac{30(s-3)}{[(s-3)^2+4][(s-3)^2+64]}$

Thus $\mathbf{L[e^{3t}\sin 5t \sin 3t] = \frac{30(s-3)}{(s^2-6s+13)(s^2-6s+73)}}$

---

20. Let $f(t) = \sinh at \sin at = \frac{e^{at} - e^{-at}}{2} \cdot \sin at$

ie., $f(t) = \frac{1}{2}(e^{at}\sin at - e^{-at}\sin at)$

$\therefore \quad L[f(t)] = \frac{1}{2}\{L(\sin at)_{s \to s-a} - L(\sin at)_{s \to s+a}\}$

But $L(\sin at) = \frac{a}{s^2+a^2}$

Hence $L(\sinh at \sin at) = \frac{1}{2}\left\{\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right\}$

$$= \frac{a}{2}\left\{\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right\}$$

$$= \frac{a}{2}\left\{\frac{4as}{(s^2+2a^2)^2 - 4a^2s^2}\right\}$$

Thus $\mathbf{L(\sinh at \sin at) = \frac{2a^2 s}{s^4+4a^4}}$

---

21. Let $f(t) = \sin^3 2t = \frac{1}{4}(3\sin 2t - \sin 6t)$

$$L(\sin^3 2t) = \frac{1}{4}\left(3 \cdot \frac{2}{s^2+4} - \frac{6}{s^2+36}\right) = \frac{48}{(s^2+4)(s^2+36)}$$

Now $L(\cosh t \sin^3 2t) = L\left[\frac{e^t+e^{-t}}{2} \cdot \sin^3 2t\right]$

$$= \frac{1}{2}\left\{L(e^t \sin^3 2t) + L(e^{-t}\sin^3 2t)\right\}$$

$$= \frac{1}{2}\left\{L(\sin^3 2t)_{s \to s-1} + L(\sin^3 2t)_{s \to s+1}\right\}$$

$$= \frac{1}{2}\left\{\frac{48}{[(s-1)^2+4][(s-1)^2+36]} + \frac{48}{[(s+1)^2+4][(s+1)^2+36]}\right\}$$

$$\boldsymbol{L(\cosh t \sin^3 2t) = 24\left\{\frac{1}{(s^2-2s+5)(s^2-2s+37)} + \frac{1}{(s^2+2s+5)(s^2+2s+37)}\right\}}$$

Find the Laplace transform of the following functions

22. $t\cos at$	23. $t^2 \sin at$	24. $t^3 \sin t$
25. $t^3 \cosh t$	26. $t^5 e^{4t} \cosh 3t$	27. $t e^{-2t} \sin 4t$

22. Let $f(t) = \cos at \;\therefore\; L[f(t)] = \frac{s}{s^2+a^2}$

Now $L[tf(t)] = \frac{-d}{ds}\left(\frac{s}{s^2+a^2}\right) = -\left\{\frac{(s^2+a^2)1 - s \cdot 2s}{(s^2+a^2)^2}\right\}$

Hence $\boldsymbol{L(t\cos at) = \frac{s^2-a^2}{(s^2+a^2)^2}}$

---

23. Let $f(t) = \sin at \;\therefore\; L[f(t)] = \frac{a}{s^2+a^2}$

Now $L[t^2 f(t)] = \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) = \frac{d}{ds}\left\{\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)\right\}$

$$L[t^2 \sin at] = \frac{d}{ds}\left\{\frac{-2as}{(s^2+a^2)^2}\right\}$$

$$L[t^2 \sin at] = \frac{(s^2+a^2)^2(-2a) + 2as \cdot 2(s^2+a^2) \cdot 2s}{(s^2+a^2)^4}$$

$$= \frac{2a(s^2+a^2)\{-(s^2+a^2)+4s^2\}}{(s^2+a^2)^4}$$

Thus $\quad \mathbf{L[t^2 \sin at] = \dfrac{2a(3s^2-a^2)}{(s^2+a^2)^3}}$

---

24. Let $f(t) = \sin t \quad \therefore \quad L[f(t)] = \dfrac{1}{s^2+1}$

Now $\quad L[t^3 \sin t] = (-1)^3 \dfrac{d^3}{ds^3}\left(\dfrac{1}{s^2+1}\right) = \dfrac{-d}{ds} \cdot \dfrac{d}{ds}\left\{\dfrac{d}{ds}\left(\dfrac{1}{s^2+1}\right)\right\}$

$$L[t^3 \sin t] = \frac{-d}{ds}\frac{d}{ds}\left\{\frac{-2s}{(s^2+1)^2}\right\} = \frac{d}{ds}\left\{\frac{(s^2+1)^2 \cdot 2 - 2s \cdot 2(s^2+1) \cdot 2s}{(s^2+1)^4}\right\}$$

$$L[t^3 \sin t] = \frac{d}{ds}\left\{\frac{2(s^2+1)[s^2+1-4s^2]}{(s^2+1)^4}\right\} = 2\frac{d}{ds}\left\{\frac{1-3s^2}{(s^2+1)^3}\right\}$$

$$L[t^3 \sin t] = 2\left\{\frac{(s^2+1)^3(-6s) - (1-3s^2) \cdot 3(s^2+1)^2 \cdot 2s}{(s^2+1)^6}\right\}$$

$$= -12s(s^2+1)^2\left\{\frac{(s^2+1)+(1-3s^2)}{(s^2+1)^6}\right\}$$

Thus $\quad \mathbf{L[t^3 \sin t] = \dfrac{24s(s^2-1)}{(s^2+1)^4}}$

---

25. Let $f(t) = t^3 \cosh t$

Note : *Here we should not prefer to work the problem similar to the previous problem as we have* $\cosh t$ *which can be converted to the form* $(e^t + e^{-t})/2$ *so that it will be highly convenient to apply the shifting property.*

$$f(t) = t^3\left(\frac{e^t+e^{-t}}{2}\right) = \frac{1}{2}\{e^t t^3 + e^{-t} t^3\}$$

$$L[f(t)] = \frac{1}{2}\left\{L(t^3)_{s \to s-1} + L(t^3)_{s \to s+1}\right\}.$$

But $L(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$

Thus $L[t^3 \cosh t] = \frac{1}{2}\left\{\frac{6}{(s-1)^4} + \frac{6}{(s+1)^4}\right\} = 3\left\{\frac{1}{(s-1)^4} + \frac{1}{(s+1)^4}\right\}$

---

26. Let $f(t) = t^5 e^{4t} \cosh 3t = t^5 e^{4t} \cdot \frac{1}{2}(e^{3t} + e^{-3t})$

ie., $f(t) = \frac{1}{2}(e^{7t} t^5 + e^t t^5)$

$L[f(t)] = \frac{1}{2}\left\{L(t^5)_{s \to s-7} + L(t^5)_{s \to s-1}\right\}$

But $L(t^5) = \frac{5!}{s^6} = \frac{120}{s^6}$

Thus $L(t^5 e^{4t} \cosh 3t) = \frac{1}{2}\left\{\frac{120}{(s-7)^6} + \frac{120}{(s-1)^6}\right\} = 60\left\{\frac{1}{(s-7)^6} + \frac{1}{(s-1)^6}\right\}$

---

27. Let $f(t) = t e^{-2t} \sin 4t$

$L(\sin 4t) = \frac{4}{s^2+16} \quad \therefore \quad L(e^{-2t} \sin 4t) = \frac{4}{(s+2)^2+16} = \frac{4}{s^2+4s+20}$

Hence $L(t e^{-2t} \sin 4t) = \frac{-d}{ds}\left\{\frac{4}{s^2+4s+20}\right\} = \frac{4(2s+4)}{(s^2+4s+20)^2}$

Thus $L(t e^{-2t} \sin 4t) = \frac{8(s+2)}{(s^2+4s+20)^2}$

---

28. *Show that* $\int_0^\infty t^3 e^{-t} \sin t \, dt = 0$

\>> We have $\int_0^\infty e^{-st} \cdot t^3 \sin t \, dt = L(t^3 \sin t)$

Referring to Problem-24 for the R.H.S we have

$$\int_0^\infty e^{-st} \cdot t^3 \sin t \, dt = \frac{24s(s^2-1)}{(s^2+1)^4}$$

Thus by putting $s=1$ we get $\int_0^\infty e^{-t} t^3 \sin t \, dt = 0$

---

29. *Show that* $\int_0^\infty t\,e^{-2t}\sin 4t\,dt = \frac{1}{25}$

>> We have $\int_0^\infty e^{-st}\,t\sin 4t\,dt = L(t\sin 4t)$ ... (1)

Now $L(t\sin 4t) = \frac{-d}{ds}L(\sin 4t) = \frac{-d}{ds}\left(\frac{4}{s^2+16}\right) = \frac{8s}{(s^2+16)^2}$

Hence (1) becomes $\int_0^\infty e^{-st}\,t\sin 4t\,dt = \frac{8s}{(s^2+16)^2}$

Thus by putting $s=2$ we get, $\int_0^\infty e^{-2t}\,t\sin 4t\,dt = \frac{16}{400} = \frac{1}{25}$

---

30. *Find the value of* $\int_0^\infty t\,e^{-3t}\cos 2t\,dt$ *using Laplace transforms.*

>> We have $\int_0^\infty e^{-st}\,t\cos 2t\,dt = L(t\cos 2t)$ ... (1)

Proceeding as in Problem-22 we can obtain $L(t\cos 2t) = \frac{s^2-4}{(s^2+4)^2}$

Using this result in the R.H.S of (1) we have,

$$\int_0^\infty e^{-st}\,t\cos 2t\,dt = \frac{s^2-4}{(s^2+4)^2}$$

Thus by putting $s=3$ we get $\int_0^\infty e^{-3t}\,t\cos 2t\,dt = \frac{5}{169}$

Find the Laplace transform of the following functions.

31. $\frac{1-e^{-at}}{t}$
32. $\frac{\cos at - \cos bt}{t}$
33. $\frac{\sinh t}{t}$
34. $\frac{\sin^2 t}{t}$
35. $\frac{2\sin t\sin 5t}{t}$
36. $\frac{\sin at}{t}$

31. Let $f(t) = 1 - e^{-at}$ $\therefore$ $\bar{f}(s) = L[f(t)] = \frac{1}{s} - \frac{1}{s+a}$

>> We have $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\, ds$

Hence $L\left[\frac{1-e^{-at}}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+a}\right) ds$

$$= [\log s - \log(s+a)]_s^\infty = \left[\log\left(\frac{s}{s+a}\right)\right]_s^\infty$$

ie., $$= \lim_{s\to\infty} \log\left(\frac{s}{s+a}\right) - \log\left(\frac{s}{s+a}\right)$$

$$= \lim_{s\to\infty} \log\left(\frac{\not s}{\not s(1+a/s)}\right) - \log\left(\frac{s}{s+a}\right) = \log 1 - \log\left(\frac{s}{s+a}\right)$$

Thus $\mathbf{L\left[\frac{1-e^{-at}}{t}\right] = \log\left(\frac{s+a}{s}\right)}$

---

32. Let $f(t) = \cos at - \cos bt$ $\therefore$ $\bar{f}(s) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

Hence $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty = \frac{1}{2}\left\{\lim_{s\to\infty} \log\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$$

$$= \frac{1}{2}\left\{\log 1 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\} = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Thus $\mathbf{L\left[\frac{\cos at - \cos bt}{t}\right] = \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}}$

---

33. Let $f(t) = \sinh t \quad \therefore \quad \bar{f}(s) = \dfrac{1}{s^2-1}$

Hence $L\left[\dfrac{\sinh t}{t}\right] = \displaystyle\int_s^\infty \frac{1}{s^2-1}\, ds = \frac{1}{2}\left[\log\left(\frac{s-1}{s+1}\right)\right]_s^\infty$

ie.,
$$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{s(1-1/s)}{s(1+1/s)}\right] - \frac{1}{2}\log\left(\frac{s-1}{s+1}\right)$$

$$= \frac{1}{2}\left\{\log 1 - \log\left(\frac{s-1}{s+1}\right)\right\} = \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

Thus $\quad \boldsymbol{L\left[\dfrac{\sin ht}{t}\right] = \log\sqrt{\dfrac{s+1}{s-1}}}$

---

34. Let $f(t) = \sin^2 t = \dfrac{1}{2}(1-\cos 2t)$

$\therefore \quad \bar{f}(s) = \dfrac{1}{2}\left(\dfrac{1}{s} - \dfrac{s}{s^2+4}\right)$

Hence $L\left[\dfrac{f(t)}{t}\right] = \dfrac{1}{2}\displaystyle\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$

ie., $\quad L\left[\dfrac{\sin^2 t}{t}\right] = \dfrac{1}{2}\left[\log s - \dfrac{1}{2}\log(s^2+4)\right]_s^\infty = \dfrac{1}{2}\log\left[\dfrac{s}{\sqrt{s^2+4}}\right]_s^\infty$

ie.,
$$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{s}{s\sqrt{1+(4/s^2)}}\right] - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}\left\{\log 1 - \log\left(\frac{s}{\sqrt{s^2+4}}\right)\right\}$$

Thus $\quad \boldsymbol{L\left[\dfrac{\sin^2 t}{t}\right] = \dfrac{1}{2}\log\left(\dfrac{\sqrt{s^2+4}}{s}\right)}$

---

35. Let $f(t) = 2\sin t \sin 5t = 2 \cdot \frac{1}{2}[\cos(-4t) - \cos(6t)]$

ie., $f(t) = \cos 4t - \cos 6t \qquad \therefore \quad \bar{f}(s) = \frac{s}{s^2+16} - \frac{s}{s^2+36}$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+16} - \frac{s}{s^2+36}\right) ds = \frac{1}{2}\Big[\log(s^2+16) - \log(s^2+36)\Big]_s^\infty$$

$$L\left[\frac{2\sin t\sin 5t}{t}\right] = \frac{1}{2}\left[\log\left(\frac{s^2+16}{s^2+36}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty} \frac{1}{2}\cdot\log\left[\frac{s^2(1+16/s^2)}{s^2(1+36/s^2)}\right] - \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$$

$$= \frac{1}{2}\left\{\log 1 - \log\left(\frac{s^2+16}{s^2+36}\right)\right\} = \frac{1}{2}\log\left(\frac{s^2+36}{s^2+16}\right)$$

Thus $\displaystyle L\left[\frac{2\sin t\,\sin 5t}{t}\right] = \log\sqrt{\frac{s^2+36}{s^2+16}}$

- -

36. Let $f(t) = \sin at \quad \therefore \quad \bar{f}(s) = \frac{a}{s^2+a^2}$

Hence $\displaystyle L\left[\frac{f(t)}{t}\right] = \int_s^\infty \frac{a}{s^2+a^2}\,ds$

ie., $\displaystyle L\left[\frac{\sin at}{t}\right] = a\cdot\frac{1}{a}\Big[\tan^{-1}(s/a)\Big]_s^\infty = \tan^{-1}(\infty) - \tan^{-1}(s/a)$

Thus $\displaystyle L\left[\frac{\sin at}{t}\right] = \pi/2 - \tan^{-1}(s/a) = \cot^{-1}(s/a)$

- -

Evaluate the following integrals using Laplace transforms.

37. $\displaystyle\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt$ 38. $\displaystyle\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\,dt$ 39. $\displaystyle\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt$

37. We know that $\int_0^\infty e^{-st}\left[\frac{\cos 6t-\cos 4t}{t}\right]dt = L\left[\frac{\cos 6t-\cos 4t}{t}\right]$

Proceeding as in Problem-32 we can obtain

$$L\left[\frac{\cos 6t-\cos 4t}{t}\right] = \log\sqrt{(s^2+16)/(s^2+36)}$$

ie.,
$$\int_0^\infty e^{-st}\left[\frac{\cos 6t-\cos 4t}{t}\right]dt = \log\sqrt{(s^2+16)/(s^2+36)}$$

Thus by putting $s=0$ we get,

$$\int_0^\infty \frac{\cos 6t-\cos 4t}{t}\,dt = \log\sqrt{16/36} = \log(2/3)$$

---

38. We shall first find $L\left[\frac{e^{-at}-e^{-bt}}{t}\right]$

Now, $L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty [L(e^{-at})-L(e^{-bt})]\,dt$

$$= \int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds = \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

ie.,
$$= \lim_{s\to\infty}\log\left[\frac{s(1+a/s)}{s(1+b/s)}\right]-\log\left(\frac{s+a}{s+b}\right) = \log 1-\log\left(\frac{s+a}{s+b}\right)$$

$$\therefore \quad L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \log\left(\frac{s+b}{s+a}\right)$$

ie.,
$$\int_0^\infty e^{-st}\left[\frac{e^{-at}-e^{-bt}}{t}\right]dt = \log\left(\frac{s+b}{s+a}\right)$$

Thus by putting $s=0$ we get, $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt = \log(b/a)$

---

39. We shall first find $L\left[\frac{e^{-t}\sin t}{t}\right]$

Now $L\left[\frac{e^{-t}\sin t}{t}\right] = \int_s^{\infty} L(e^{-t}\sin t)\,ds = \int_s^{\infty}\left\{L(\sin t)_{s\to s+1}\right\}ds$

ie., $= \int_s^{\infty}\frac{1}{(s+1)^2+1}\,ds = \left[\tan^{-1}(s+1)\right]_s^{\infty} = \tan^{-1}(\infty) - \tan^{-1}(s+1)$

$\therefore \quad L\left[\frac{e^{-t}\sin t}{t}\right] = \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$

ie., $\int_0^{\infty} e^{-st}\cdot\frac{e^{-t}\sin t}{t}\,dt = \cot^{-1}(s+1)$

Thus by putting $s = 0$ we get, $\boldsymbol{\int_0^{\infty}\frac{e^{-t}\sin t}{t}\,dt = \cot^{-1}(1) = \frac{\pi}{4}}$

---

40. *Find* $L\left[\frac{\sin^2 t}{t^2}\right]$

We know that $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty}\bar{f}(s)\,ds$

Further we have $L\left[\frac{f(t)}{t^2}\right] = \int_s^{\infty}\left[\int_s^{\infty}\bar{f}(s)\,ds\right]ds$

$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$ *(Refer Problem-34)*

$\therefore \quad L\left[\frac{\sin^2 t}{t^2}\right] = \int_s^{\infty}\frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)ds = \int_s^{\infty}\frac{1}{2}\log\sqrt{\frac{s^2+4}{s^2}}\,ds$

ie., $= \frac{1}{4}\int_s^{\infty}\log\left(1+\frac{4}{s^2}\right)ds = \frac{1}{4}\int_s^{\infty}\log\left(1+\frac{4}{s^2}\right)\cdot 1\,ds$

Integrating by parts we have,

$$L\left[\frac{\sin^2 t}{t^2}\right] = \frac{1}{4}\left\{\left[\log\left(1+\frac{4}{s^2}\right)\cdot s\right]_s^{\infty} - \int_s^{\infty} s\cdot\frac{1}{1+(4/s^2)}\cdot\left(\frac{-8}{s^3}\right) ds\right\}$$

$$= \frac{1}{4}\left\{\left[0-\log\left(1+\frac{4}{s^2}\right)s\right]+\int_s^{\infty}\frac{8}{s^2+4}\,ds\right\}$$

$$= \frac{-s}{4}\log\left(\frac{s^2+4}{s^2}\right)+\left[\tan^{-1}(s/2)\right]_s^{\infty}$$

$$= \frac{s}{4}\log\left(\frac{s^2}{s^2+4}\right) + [\pi/2-\tan^{-1}(s/2)]$$

Thus $$L\left[\frac{\sin^2 t}{t^2}\right] = \frac{s}{4}\log\left(\frac{s^2}{s^2+4}\right)+\cot^{-1}\left(\frac{s}{2}\right)$$

Find the Laplace transform of the following functions

41. $\int_0^t \sinh at \sin at\, dt$

42. $\int_0^t t\cos at\, dt$

43. $\int_0^t e^{2t}\frac{\sin at}{t}\, dt$

44. $e^{-4t}\int_0^t t\sin 3t\, dt$

41. Let $f(t) = \sinh at \sin at$

$\therefore \quad \bar{f}(s) = \dfrac{2a^2 s}{s^4+4a^4}$ *(Refer Problem-20)*

We have $$L\left[\int_0^t f(t)\,dt\right] = \frac{\bar{f}(s)}{s}$$

Thus $$L\left[\int_0^t \sinh at\sin at\,dt\right] = \frac{2a^2}{s^4+4a^4}$$

---

42. Let $f(t) = t\cos at \;\therefore\; \bar{f}(s) = \dfrac{s^2 - a^2}{(s^2+a^2)^2}$ *(Refer Problem-22)*

We have $L\left[\int_0^t f(t)\,dt\right] = \dfrac{\bar{f}(s)}{s}$

Thus $L\left[\int_0^t t\cos at\,dt\right] = \dfrac{s^2 - a^2}{s(s^2+a^2)^2}$

---

43. Let $f(t) = e^{2t}\cdot\dfrac{\sin at}{t}$

$\therefore \quad L[f(t)] = L\left[\dfrac{\sin at}{t}\right]_{s\to(s-2)}$

But $L\left[\dfrac{\sin at}{t}\right] = \cot^{-1}(s/a)$ *(Refer Problem-36)*

Hence $L\left[e^{2t}\cdot\dfrac{\sin at}{t}\right] = \cot^{-1}\left(\dfrac{s-2}{a}\right) = \bar{f}(s)$

But $L\left[\int_0^t f(t)\,dt\right] = \dfrac{\bar{f}(s)}{s}$

Thus $L\left[\int_0^t e^{2t}\,\dfrac{\sin at}{t}\,dt\right] = \dfrac{1}{s}\cot^{-1}\left(\dfrac{s-2}{a}\right)$

---

44. To find $L\left[e^{-4t}\int_0^t t\sin 3t\,dt\right]$ we shall first find $L(t\sin 3t)$

$$L(t\sin 3t) = \frac{-d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

$$\therefore \quad L\left[\int_0^t t\sin 3t\,dt\right] = \frac{1}{s}\cdot\frac{6s}{(s^2+9)^2} = \frac{6}{(s^2+9)^2}$$

Thus $L\left[e^{-4t}\int_0^t t\sin 3t\,dt\right] = \dfrac{6}{[(s+4)^2+9]^2} = \dfrac{6}{(s^2+8s+25)^2}$

MISCELLANEOUS PROBLEMS

45. *Find the Laplace transform of* $2^t + \dfrac{\cos 2t - \cos 3t}{t} + t \sin t$

>> The given function be denoted by $f(t)$ and let

$$f(t) = F(t) + G(t) + H(t)$$

where $F(t) = 2^t$, $G(t) = \dfrac{\cos 2t - \cos 3t}{t}$, $H(t) = t\sin t$

$$\therefore \quad L[f(t)] = L[F(t)] + L[G(t)] + L[H(t)] \qquad \ldots (1)$$

Now $\quad L[F(t)] = L[2^t] = L[e^{\log 2 \cdot t}] = \dfrac{1}{s - \log 2}$

Further $G(t) = \dfrac{\cos 2t - \cos 3t}{t}$

$$\therefore \quad L[G(t)] = \int_s^\infty L(\cos 2t - \cos 3t)\, ds$$

$$= \int_s^\infty \left[\frac{s}{s^2+4} - \frac{s}{s^2+9} \right] ds$$

$$= \left[\frac{1}{2}\log(s^2+4) - \frac{1}{2}\log(s^2+9) \right]_s^\infty$$

$$= \left[\log \sqrt{s^2+4/s^2+9} \right]_s^\infty$$

$$= \left[\log \sqrt{1+(4/s^2)/1+(9/s^2)} \right]_{s=\infty} - \log \sqrt{s^2+4/s^2+9}$$

$$= \log 1 - \log \sqrt{s^2+4/s^2+9}$$

ie., $\quad L[G(t)] = \log \sqrt{s^2+9/s^2+4}$

Further $H(t) = t \sin t$

$$\therefore \quad L[H(t)] = -\frac{d}{ds} L(\sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

Hence $L[H(t)] = \dfrac{2s}{(s^2+1)^2}$

Thus the required $L[f(t)]$ is given by

$$\frac{1}{s - \log 2} + \log \sqrt{s^2+9/s^2+4} + \frac{2s}{(s^2+1)^2}$$

Find the Laplace transform of $t^2 e^{-3t} \sin 2t$

>> We shall first find $L(t^2 \sin 2t)$

We have $L(t^2 \sin 2t) = (-1)^2 \dfrac{d^2}{ds^2} L(\sin 2t)$

ie., $$L(t^2 \sin 2t) = \frac{d}{ds} \cdot \frac{d}{ds}\left[\frac{2}{s^2+4}\right]$$

$$= \frac{d}{ds}\left[\frac{-4s}{(s^2+4)^2}\right]$$

$$= \frac{(s^2+4)^2(-4) + 4s \cdot 2\,(s^2+4)\,2s}{(s^2+4)^4}$$

$$= \frac{4(s^2+4)[-(s^2+4)+4s^2]}{(s^2+4)^4}$$

ie., $$L(t^2 \sin 2t) = \frac{4(3s^2-4)}{(s^2+4)^3}$$

Thus $$\boldsymbol{L(e^{-3t}\, t^2 \sin 2t) = \frac{4[3(s+3)^2-4]}{[(s+3)^2+4]^3}}$$

---

42. Find the Laplace transform of (i) $te^{2t} - \dfrac{2\sin 3t}{t}$ (ii) $\displaystyle\int_0^t e^{-t} \sin 2t \sin 3t \, dt$

(i) Let $f(t) = t\,e^{2t} - \dfrac{2\sin 3t}{t} = f_1(t) - f_2(t)$ *(say)*

$\therefore$ $$L[f(t)] = L[f_1(t)] - L[f_2(t)]$$

Now $$L[f_1(t)] = L(te^{2t}) = \{L(t)\}_{s \to s-2} = \left\{\frac{1}{s^2}\right\}_{s \to s-2}$$

$$L[f_1(t)] = \frac{1}{(s-2)^2}$$

Next $$L[f_2(t)] = 2L\left[\frac{\sin 3t}{t}\right]$$

$$= 2\int_s^\infty L(\sin 3t)\,ds = 2\int_s^\infty \frac{3}{s^2+3^2}\,ds$$

$$L[f_2(t)] = 2[\tan^{-1}(s/3)]_s^{\infty} = 2\left\{\pi/2 - \tan^{-1}(s/3)\right\} = 2\cot^{-1}(s/3)$$

$$L[f_2(t)] = 2\cot^{-1}(s/3)$$

Hence the required $\mathbf{L[f(t)] = \dfrac{1}{(s-2)^2} - 2\cot^{-1}(s/3)}$

(ii) We have $L\left[\displaystyle\int_0^t f(t)\,dt\right] = \dfrac{\overline{f}(s)}{s}$ where $L[f(t)] = \overline{f}(s)$

Taking $f(t) = e^{-t}\sin 2t \sin 3t$ we have

$$\sin 2t \sin 3t = \frac{1}{2}\left\{\cos(2t-3t) - \cos(2t+3t)\right\} = \frac{1}{2}(\cos t - \cos 5t)$$

$$L(\sin 2t \sin 3t) = \frac{1}{2}\left\{\frac{s}{s^2+1} - \frac{s}{s^2+25}\right\} = \frac{12s}{(s^2+1)(s^2+25)}$$

$$L(e^{-t}\sin 2t \sin 3t) = \frac{12(s+1)}{[(s+1)^2+1][(s+1)^2+25]} = \overline{f}(s)$$

Thus $\mathbf{L\left[\displaystyle\int_0^t e^{-t}\sin 2t \sin 3t\,dt\right] = \dfrac{12(s+1)}{s(s^2+2s+2)(s^2+2s+26)}}$

---

48. (i) Evaluate $L\ t(\sin^3 t - \cos^3 t)$

\>> $$\sin^3 t - \cos^3 t = \frac{1}{4}(3\sin t - \sin 3t) - \frac{1}{4}(3\cos t + \cos 3t)$$

$$L(\sin^3 t - \cos^3 t) = \frac{1}{4}\left\{\frac{3}{s^2+1} - \frac{3}{s^2+9}\right\} - \frac{1}{4}\left\{\frac{3s}{s^2+1} + \frac{s}{s^2+9}\right\}$$

Using the property : $L[tf(t)] = -\dfrac{d}{ds}[\overline{f}(s)]$ we have,

$$L\left\{t(\sin^3 t - \cos^3 t)\right\} = -\frac{3}{4}\left[\frac{-2s}{(s^2+1)^2} + \frac{2s}{(s^2+9)^2}\right]$$
$$+ \frac{1}{4}\left[3\cdot\frac{(s^2+1)-2s^2}{(s^2+1)^2} + \frac{(s^2+9)-2s^2}{(s^2+9)^2}\right]$$

Thus $\mathbf{L\left\{t(\sin^3 t - \cos^3 t)\right\}}$

$$= \frac{3s}{2}\left[\frac{1}{(s^2+1)^2} - \frac{1}{(s^2+9)^2}\right] + \frac{1}{4}\left[3\cdot\frac{(1-s^2)}{(s^2+1)^2} + \frac{9-s^2}{(s^2+9)^2}\right]$$

---

49. *Using Laplace transforms evaluate* $\int_0^\infty e^{-t}\, t \sin^2 3t\, dt$

>> We shall first find $L\,(t \sin^2 3t)$

$$\sin^2 3t = \frac{1}{2}(1 - \cos 6t)$$

$$L(\sin^2 3t) = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$$

$$L(t \sin^2 3t) = \frac{1}{2} \cdot -\frac{d}{ds}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{s^2} - \frac{(s^2+36) - 2s^2}{(s^2+36)^2}\right]$$

$$\therefore \quad L(t \sin^2 3t) = \frac{1}{2}\left[\frac{1}{s^2} + \frac{(36 - s^2)}{(s^2+36)^2}\right]$$

Using the basic definition in L.H.S we have,

$$\int_0^\infty e^{-st}\, t \sin^2 3t\, dt = \frac{1}{2}\left[\frac{1}{s^2} + \frac{36 - s^2}{(s^2+36)^2}\right]$$

Thus by putting $s = 1$ we get,

$$\int_0^\infty e^{-t}\, t \sin^2 3t\, dt = \frac{1}{2}\left[1 + \frac{35}{(37)^2}\right] = \mathbf{\frac{702}{1369}}$$

---

50. *Find the Laplace transform of* (i) $e^{2t}\cos^2 t$ (ii) $\frac{1-\cos 3t}{t}$

(i) Let $f(t) = e^{2t}\cos^2 t = e^{2t} \cdot \frac{1}{2}(1 + \cos 2t)$

$$L[f(t)] = \frac{1}{2}\left[L(e^{2t}) + L(e^{2t}\cos 2t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-2} + \left\{L(\cos 2t)\right\}_{s \to s-2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-2} + \left\{\frac{s}{s^2+4}\right\}_{s \to s-2}\right]$$

Thus $L(e^{2t}\cos^2 t) = \frac{1}{2}\left[\frac{1}{s-2} + \frac{s-2}{s^2-4s+8}\right]$

(ii) Let $f(t) = 1 - \cos 3t$

$\therefore \quad \bar{f}(s) = L[f(t)] = \frac{1}{s} - \frac{s}{s^2+9}$

We have the property : $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$

i.e., $L\left[\frac{1-\cos 3t}{t}\right] = \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+9}\right] ds$

$$= \left[\log s - \frac{1}{2}\log(s^2+9)\right]_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2+9}}\right]_s^\infty$$

$$= \left[\log \frac{s}{s\sqrt{1+9/s^2}}\right]_{s=\infty} - \log \frac{s}{\sqrt{s^2+9}}$$

$$= \log 1 - \log \frac{s}{\sqrt{s^2+9}} = \log\left[\frac{\sqrt{s^2+9}}{s}\right]$$

Thus $L\left[\frac{1-\cos 3t}{t}\right] = \log\left[\frac{\sqrt{s^2+9}}{s}\right]$

---

51. *Find* $L[\sin\sqrt{t}]$

>> We have the expansion of $\sin x$ given by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\therefore \quad \sin(\sqrt{t}) = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots$$

$$L[\sin(\sqrt{t})] = L(t^{1/2}) - \frac{L(t^{3/2})}{6} + \frac{L(t^{5/2})}{120} \qquad \ldots (1)$$

$$L(t^{1/2}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{1/2 \cdot \Gamma(1/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$L(t^{3/2}) = \frac{\Gamma(5/2)}{s^{5/2}} = \frac{3/2 \cdot 1/2 \cdot \sqrt{\pi}}{s^{5/2}} = \frac{3\sqrt{\pi}}{4\,s^{5/2}}$$

$$L(t^{5/2}) = \frac{\Gamma(7/2)}{s^{7/2}} = \frac{5/2 \cdot 3/2 \cdot 1/2 \cdot \sqrt{\pi}}{s^{7/2}} = \frac{15\sqrt{\pi}}{8\,s^{7/2}}$$

Substituting these values in (1) we get,

$$L[\sin\sqrt{t}\,] = \frac{\sqrt{\pi}}{2\,s^{3/2}} - \frac{\sqrt{\pi}}{8\,s^{5/2}} + \frac{\sqrt{\pi}}{64\,s^{7/2}} - \cdots$$

$$= \frac{\sqrt{\pi}}{2\,s^{3/2}}\left\{1 - \frac{1}{4\,s} + \frac{1}{32\,s^2} - \cdots\right\}$$

$$= \frac{\sqrt{\pi}}{2\,s^{3/2}}\left\{1 - \frac{1/4\,s}{1!} + \frac{(1/4s)^2}{2!} - \cdots\right\} = \frac{\sqrt{\pi}}{2s^{3/2}}\,e^{-1/4s}$$

Thus $\quad \boldsymbol{L[\sin\sqrt{t}\,] = \frac{1}{2s}\sqrt{\frac{\pi}{s}}\;e^{-1/4\,s}}$

---

(52) *Find* $L\left[\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right]$

>> We have $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$

$$\therefore \quad \frac{\cos(\sqrt{t}\,)}{\sqrt{t}} = \frac{1}{\sqrt{t}}\left\{1 - \frac{t}{2!} + \frac{t^2}{4!} - \cdots\right\}$$

ie.,
$$\frac{\cos(\sqrt{t}\,)}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2} + \frac{t^{3/2}}{24} - \cdots$$

Now
$$L\left[\frac{\cos(\sqrt{t}\,)}{\sqrt{t}}\right] = L(t^{-1/2}) - \frac{L(t^{1/2})}{2} + \frac{L(t^{3/2})}{24} - \cdots \qquad \ldots (1)$$

$$L(t^{-1/2}) = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

The expression for $L(t^{3/2})$, $L(t^{5/2})$, is as in the previous example. Substituting these in (1) we have,

$$L\left[\frac{\cos(\sqrt{t}\,)}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{s}} - \frac{1}{2}\,\frac{\sqrt{\pi}}{2s\,\sqrt{s}} + \frac{3\sqrt{\pi}}{24 \cdot 4\,s^2\sqrt{s}} - \cdots$$

ie.,
$$= \sqrt{\frac{\pi}{s}}\left\{1 - \frac{1}{4s} + \frac{1}{32s^2} - \cdots\right\} = \sqrt{\frac{\pi}{s}}\left\{1 - \frac{1/4\,s}{1!} + \frac{(1/4s)^2}{2!} - \cdots\right\}$$

Thus $\quad \boldsymbol{L\left[\frac{\cos(\sqrt{t}\,)}{\sqrt{t}}\right] = \sqrt{\pi/s}\;e^{-1/4s}}$

---

53. *If* $L[f(t)] = \bar{f}(s)$ *prove that* $L[f(at)] = 1/a \cdot \bar{f}(s/a)$

>> By definition $L[f(at)] = \int_0^\infty e^{-st} f(at)\,dt$

Put $at = u$ $\therefore$ $dt = du/a$ and u also varies from 0 to ∞

$$\therefore \quad L[f(at)] = \int_{u=0}^{\infty} e^{-su/a} f(u) \frac{du}{a} = \frac{1}{a} \int_0^\infty e^{-(s/a)u} f(u)\,du$$

Thus $\quad L[f(at)] = 1/a \cdot \bar{f}(s/a)$

Remark : *This is called 'change of scale property'*

---

54. *Prove the following :*

(i) $L\left[\frac{1}{2a}(\sin at + at\cos at)\right] = \frac{s^2}{(s^2+a^2)^2}$

(ii) $L\left[\frac{1}{2a^3}(\sin at - at\cos at)\right] = \frac{1}{(s^2+a^2)^2}$

>> We have to first find $L(at\cos at) = a\,L(t\cos at)$

Referring to Problem-22, $L(at\cos at) = \dfrac{a(s^2-a^2)}{(s^2+a^2)^2}$

Also we know that $L(\sin at) = \dfrac{a}{s^2+a^2}$

Hence $L(\sin at + at\cos at) = \dfrac{a}{s^2+a^2} + \dfrac{a(s^2-a^2)}{(s^2+a^2)^2}$

$$= a\left[\frac{s^2+a^2+s^2-a^2}{(s^2+a^2)^2}\right] = \frac{2as^2}{(s^2+a^2)^2}$$

Thus $\quad L\left[\dfrac{1}{2a}(\sin at + at\cos at)\right] = \dfrac{s^2}{(s^2+a^2)^2}$

Also $\quad L(\sin at - at\cos at) = a\left[\dfrac{(s^2+a^2)-(s^2-a^2)}{(s^2+a^2)^2}\right] = \dfrac{2a^3}{(s^2+a^2)^2}$

Thus $\quad L\left[\dfrac{1}{2a^3}(\sin at - at\cos at)\right] = \dfrac{1}{(s^2+a^2)^2}$

---

55. *Given* $L[2\sqrt{t/\pi}] = 1/s^{3/2}$ *show that* $L[1/\sqrt{\pi t}] = 1/\sqrt{s}$

>> $L[2\sqrt{t/\pi}] = 1/s^{3/2}$ by data.

$$\therefore \quad L\left[\frac{2\sqrt{t/\pi}}{t}\right] = \int_s^{\infty}\frac{1}{s^{3/2}}\,ds = \int_s^{\infty}s^{-3/2}\,ds$$

$$ie., \quad L\left[\frac{2}{\sqrt{\pi t}}\right] = \left[\frac{s^{-1/2}}{-1/2}\right]_s^{\infty} = \left[\frac{-2}{\sqrt{s}}\right]_s^{\infty} = \frac{2}{\sqrt{s}}$$

$$\text{Thus} \quad L\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{s}}$$

7.6 Laplace transform of periodic function

Definition : A function $f(t)$ is said to be a periodic function of period $T > 0$ if $f(t+nT) = f(t)$ where $n = 1, 2, 3 \cdots$

Example : $\sin t$, $\cos t$ are periodic functions of period 2π because

$$\sin(t+2n\pi) = \sin t, \quad \cos(t+2n\pi) = \cos t.$$

Theorem : *If $f(t)$ is a periodic function of period T, then*

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^{T} e^{-st}f(t)\,dt$$

Proof : We have by the definition

$$L[f(t)] = \int_0^{\infty} e^{-st}f(t)\,dt = \int_0^{\infty} e^{-su}f(u)\,du$$

$$= \int_{u=0}^{T} e^{-su}f(u)\,du + \int_{u=T}^{2T} e^{-su}f(u)\,du + \cdots + \int_{u=nT}^{(n+1)T} e^{-su}f(u)\,du + \cdots$$

$$L[f(t)] = \sum_{n=0}^{\infty}\int_{u=nT}^{(n+1)T} e^{-su}f(u)\,du$$

Now put $u = t+nT \quad \therefore \; du = dt$

If $\quad u = nT \quad$ then $t+nT = nT \Rightarrow t = 0$

$\quad u = (n+1)T \quad$ then $t+nT = nT+T \Rightarrow t = T$

Further $f(u) = f(t+nT) = f(t)$ by the periodic property.

Using these results in the R.H.S of (1) we obtain

$$L[f(t)] = \sum_{n=0}^{\infty} \int_{t=0}^{T} e^{-s(t+nT)} f(t)\, dt$$

$$L[f(t)] = \sum_{n=0}^{\infty} e^{-snT} \int_{t=0}^{T} e^{-st} f(t)\, dt \qquad \ldots (2)$$

But $\sum_{n=0}^{\infty} e^{-snT} = \sum_{n=0}^{\infty} \left(e^{-sT}\right)^n = 1 + (e^{-sT}) + (e^{-sT})^2 + \cdots$

Putting $r = e^{-sT}$ the series involved is a geometric series of the form $1 + r + r^2 + \cdots$ whose sum to infinity is known to be $1/1 - r$ where $r < 1$.

Hence $\sum_{n=0}^{\infty} e^{-snT} = \dfrac{1}{1 - e^{-sT}} \qquad \ldots (3)$

Now using (3) in the R.H.S of (2) we have

$$L[f(t)] = \frac{1}{1 - e^{-sT}} \int_{0}^{T} e^{-st} f(t)\, dt$$

WORKED PROBLEMS

56. *If $f(t) = t^2$, $0 < t < 2$ and $f(t+2) = f(t)$ for $t > 2$, find $L[f(t)]$*

>> $f(t)$ is a periodic function of period 2. $\therefore\ T = 2$

We have $L[f(t)] = \dfrac{1}{1 - e^{-sT}} \displaystyle\int_{0}^{T} e^{-st} f(t)\, dt$

Now, $L[f(t)] = \dfrac{1}{1 - e^{-2s}} \displaystyle\int_{0}^{2} e^{-st} \cdot t^2\, dt = \dfrac{1}{1 - e^{-2s}} \displaystyle\int_{0}^{2} t^2 e^{-st}\, dt$

Applying Bernoulli's rule of integration by parts,

$$L[f(t)] = \frac{1}{1 - e^{-2s}} \left\{ \left[t^2 \cdot \frac{e^{-st}}{-s} \right]_0^2 - \left[2t \cdot \frac{e^{-st}}{s^2} \right]_0^2 + \left[2 \cdot \frac{e^{-st}}{-s^3} \right]_0^2 \right\}$$

$$= \frac{1}{1 - e^{-2s}} \left\{ \frac{-1}{s} (4e^{-2s} - 0) - \frac{2}{s^2} (2e^{-2s} - 0) - \frac{2}{s^3} (e^{-2s} - 1) \right\}$$

$$L[f(t)] = \frac{2}{s^3(1-e^{-2s})}\left\{-2s^2 e^{-2s} - 2s\, e^{-2s} - e^{-2s} + 1\right\}$$

Thus $$L[f(t)] = \frac{2}{s^3(1-e^{-2s})}\left\{1 - (2s^2 + 2s + 1)\, e^{-2s}\right\}$$

---

57. *Find the Laplace transform of the periodic function defined by $f(t) = kt/T$, $0 < t < T$; $f(t+T) = f(t)$*

>> We have $$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt$$

Now, $$L[f(t)] = \frac{k}{T(1-e^{-sT})}\int_0^T t\, e^{-st}\, dt$$

Applying Bernoulli's rule we have

$$L[f(t)] = \frac{k}{T(1-e^{-sT})}\left[t \cdot \left(\frac{e^{-st}}{-s}\right) - 1 \cdot \left(\frac{e^{-st}}{s^2}\right)\right]_0^T$$

$$= \frac{k}{T(1-e^{-sT})}\left[\left(-\frac{1}{s} T e^{-sT} - 0\right) - \frac{1}{s^2}\left(e^{-sT} - 1\right)\right]$$

Thus $$L[f(t)] = \frac{-k e^{-sT}}{s(1-e^{-sT})} + \frac{k}{s^2 T}$$

---

58. *Find the Laplace transform of the full wave rectifier $f(t) = E \sin wt$, $0 < t < \pi/w$ having period π/w*

>> We have $$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt$$

Now $$L[f(t)] = \frac{1}{1-e^{-s(\pi/w)}}\int_0^{\pi/w} e^{-st} E \sin wt\, dt$$

$$= \frac{E}{1-e^{-\pi s/w}}\left[\frac{e^{-st}}{s^2+w^2}(-s \sin wt - w \cos wt)\right]_0^{\pi/w}$$

$$= \frac{E}{(1-e^{-\pi s/w})(s^2+w^2)}\left[e^{-s\pi/w} \cdot w - (-w)\right]$$

Thus $L[f(t)] = \dfrac{Ew(1+e^{-\pi s/w})}{(s^2+w^2)(1-e^{-\pi s/w})}$

Multiplying both the numerator and the denominator by $e^{\pi s/2w}$ in R.H.S, the expression assumes the form,

$$L[f(t)] = \frac{Ew}{s^2+w^2} \cdot \frac{(e^{\pi s/2w} + e^{-\pi s/2w})}{(e^{\pi s/2w} - e^{-\pi s/2w})} = \frac{Ew}{s^2+w^2} \cdot \frac{2\cosh(\pi s/2w)}{2\sinh(\pi s/2w)}$$

Thus $L[f(t)] = \dfrac{Ew}{s^2+w^2} \coth(\pi s/2w)$

---

59. *Find the Laplace transform of the function $f(t) = E\sin(\pi t/w), 0 < t < w$ given that $f(t+w) = f(t)$*

We have, $L[f(t)] = \dfrac{1}{1-e^{-sT}} \displaystyle\int_0^T e^{-st} f(t)\,dt$. Here $T = w$

Now, $L[f(t)] = \dfrac{1}{1-e^{-sw}} \displaystyle\int_0^w e^{-st} E\sin(\pi t/w)\,dt$

$$= \frac{E}{1-e^{-sw}} \left[\frac{e^{-st}}{s^2+(\pi/w)^2} \left\{ -s\sin(\pi t/w) - \frac{\pi}{w}\cos(\pi t/w) \right\} \right]_0^w$$

$$= \frac{E}{1-e^{-ws}} \cdot \frac{1}{s^2+(\pi/w)^2} \left[e^{-sw}\left(\frac{\pi}{w}\right) - 1\left(-\frac{\pi}{w}\right) \right]$$

$$= \frac{E\pi}{w(1-e^{-ws})} \cdot \frac{w^2}{w^2 s^2 + \pi^2} (1+e^{-ws})$$

$$L[f(t)] = \frac{E\pi w}{(w^2 s^2 + \pi^2)} \cdot \frac{1+e^{-ws}}{1-e^{-ws}} = \frac{E\pi w}{w^2 s^2 + \pi^2} \cdot \frac{e^{ws/2} + e^{-ws/2}}{e^{ws/2} - e^{-ws/2}}$$

Thus $L[f(t)] = \dfrac{E\pi w}{w^2 s^2 + \pi^2} \coth\left(\dfrac{ws}{2}\right)$

---

60. *Given* $f(t) = \begin{cases} E, & 0 < t < a/2 \\ -E, & a/2 < t < a \end{cases}$ *where* $f(t+a) = f(t)$, *show that* $L[f(t)] = E/s \cdot \tanh(as/4)$

The given function is periodic with period $T = a$.

We have $L[f(t)] = \dfrac{1}{1-e^{-sT}} \displaystyle\int_0^T e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-sa}} \int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}} \left\{ \int_0^{a/2} e^{-st} E\,dt + \int_{a/2}^{a} e^{-st}(-E)\,dt \right\}$$

$$= \frac{E}{1-e^{-as}} \left\{ \left[\frac{e^{-st}}{-s} \right]_0^{a/2} + \left[\frac{e^{-st}}{s} \right]_{a/2}^{a} \right\}$$

$$= \frac{E}{s(1-e^{-as})} \left\{ -\left[e^{-st} \right]_0^{a/2} + \left[e^{-st} \right]_{a/2}^{a} \right\}$$

$$= \frac{E}{s(1-e^{-as})} \left\{ -e^{-as/2} + 1 + e^{-as} - e^{-as/2} \right\}$$

$$= \frac{E}{s(1-e^{-as})} (1 - 2e^{-as/2} + e^{-as}) = \frac{E(1-e^{-as/2})^2}{s(1-e^{-as})}$$

$$L[f(t)] = \frac{E(1-e^{-as/2})^2}{s(1-e^{-as/2})(1+e^{-as/2})} = \frac{E(1-e^{-as/2})}{s(1+e^{-as/2})}$$

Multiplying both the numerator and denominator by $e^{as/4}$ we get

$$L[f(t)] = \frac{E(e^{as/4} - e^{-as/4})}{s(e^{as/4} + e^{-as/4})} = \frac{E \cdot 2\sinh(as/4)}{s \cdot 2\cosh(as/4)}$$

Thus $L[f(t)] = \dfrac{E}{s} \tan h\left(\dfrac{as}{4}\right)$

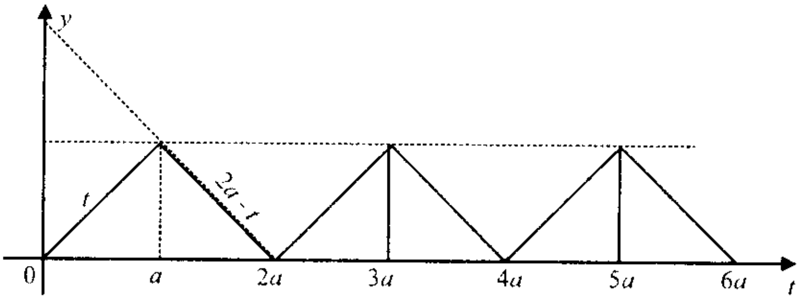
---

61. *If* $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a - t, & a \le t \le 2a, \end{cases} \quad f(t+2a) = f(t)$

(i) Sketch the graph of $f(t)$ *as a periodic function*

(ii) Show that $L[f(t)] = \dfrac{1}{s^2} \tanh(as/2)$

(i) Let $f(t) = y$ and $y = t$ is a straight line passing through the origin making an angle 45^0 with the t-axis. $y = 2a - t$ or $y + t = 2a$ or $t/2a + y/2a = 1$ is a straight line passing through the points $(2a, 0)$ and $(0, 2a)$. The graph of $y = f(t)$ is as follows.



The periodic function $f(t)$ is called the *triangular wave function.*

(ii) We have $T = 2a$ and $L[f(t)] = \dfrac{1}{1 - e^{-sT}} \displaystyle\int_0^T e^{-st} f(t)\, dt$

$$L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1 - e^{-2as}} \left\{ \int_0^a t e^{-st}\, dt + \int_a^{2a} (2a - t)\, e^{-st}\, dt \right\}$$

Applying Bernoulli's rule to each of the integrals we have,

$$L[f(t)] = \frac{1}{1 - e^{-2as}} \left\{ \left[t \cdot \frac{e^{-st}}{-s} - (1) \frac{e^{-st}}{s^2} \right]_0^a + \left[(2a - t) \frac{e^{-st}}{-s} - (-1) \frac{e^{-st}}{s^2} \right]_a^{2a} \right\}$$

$$= \frac{1}{1 - e^{-2as}} \left\{ \frac{-1}{s} (a e^{-as} - 0) - \frac{1}{s^2} (e^{-as} - 1) - \frac{1}{s} (0 - a e^{-as}) + \frac{1}{s^2} (e^{-2as} - e^{-as}) \right\}$$

$$L[f(t)] = \frac{1}{s^2(1-e^{-2as})}(-e^{-as}+1+e^{-2as}-e^{-as})$$

$$= \frac{1}{s^2(1-e^{-2as})}(1-2e^{-as}+e^{-2as}) = \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

$$L[f(t)] = \frac{(1-e^{-as})}{s^2(1+e^{-as})} = \frac{e^{as/2}-e^{-as/2}}{s^2(e^{as/2}+e^{-as/2})}$$

where we have multiplied both the numerator and denominator by $e^{as/2}$

$$\therefore \quad L[f(t)] = \frac{2\sinh(as/2)}{s^2 \cdot 2\cosh(as/2)} = \frac{1}{s^2}\tan h\left(\frac{as}{2}\right)$$

Thus $\mathbf{L[f(t)] = 1/s^2 \cdot \tanh(as/2)}$

---

62. *A periodic function of period* $2\pi/\omega$ *is defined by*

$$f(t) = \begin{cases} E\sin\omega t, & 0 \le t < \pi/\omega \\ 0, & \pi/\omega \le t < 2\pi/\omega \end{cases}$$ *where E and ω are constants.*

Show that $L[f(t)] = \dfrac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$

\>> We have for a periodic function $f(t)$,

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt. \quad \text{Here } T = 2\pi/\omega$$

$$L[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{2\pi/\omega} e^{-st}f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left\{\int_0^{\pi/\omega} e^{-st}E\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\cdot 0\,dt\right\}$$

$$= \frac{E}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{(-s)^2+\omega^2}(-s\sin\omega t-\omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{-E}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}\left\{e^{-s\pi/\omega}(s\sin\pi+\omega\cos\pi)-e^0(s\sin 0+\omega\cos 0)\right\}$$

$$= \frac{-E}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}\left\{-\omega e^{-s\pi/\omega}-\omega\right\} = \frac{E\omega(1+e^{-\pi s/\omega})}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}$$

$$L[f(t)] = \frac{E\omega(1+e^{-\pi s/\omega})}{(s^2+\omega^2)(1-e^{-\pi s/\omega})(1+e^{-\pi s/\omega})}$$

Thus $$L[f(t)] = \frac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$$

7.7 Unit step function (Heaviside function)

Definition : The ***unit step function*** $u(t-a)$ or ***Heaviside function*** $H(t-a)$ is defined as follows.

$$u(t-a) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$$ where a is a positive costant.

7.71 Properties associated with the unit step function

(i) $L[u(t-a)] = \frac{e^{-as}}{s}$

(ii) $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$ where $L[f(t)] = \bar{f}(s)$

Proof. (i) $$L[u(t-a)] = \int_0^\infty e^{-st} u(t-a)\,dt$$

$$= \int_0^a e^{-st} u(t-a)\,dt + \int_a^\infty e^{-st} u(t-a)\,dt$$

$$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\cdot 1\,dt$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty = 0 - \frac{e^{-as}}{-s} = \frac{e^{-as}}{s}$$

Thus $$L[u(t-a)] = \frac{e^{-as}}{s}$$

(ii) $L[f(t-a)u(t-a)] = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$

$= \int_0^a e^{-st} f(t-a)\cdot 0\,dt + \int_a^\infty e^{-st} f(t-a)\cdot 1\,dt$

ie., $L[f(t-a)u(t-a)] = \int_a^\infty e^{-st} f(t-a)\,dt$

Put $t-a = v$ $\therefore$ $dt = dv$ If $t = a$, $v = 0$; $t = \infty$, $v = \infty$

Hence $L[f(t-a)u(t-a)] = \int_{v=0}^\infty e^{-s(a+v)} f(v)\,dv$

ie., $= e^{-as}\int_{v=0}^\infty e^{-sv} f(v)\,dv = e^{-as}\bar{f}(s)$

Thus $\mathbf{L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)}$

Remarks

1. *The result (i) follows as a particular case of (ii) When $f(t-a) = 1$ we have $f(t)$ also equal to 1 and hence $L[f(t)] = 1/s$*

Hence (ii) becomes $L[u(t-a)] = e^{-as}/s$

2. *It is possible to express a discontinuous function $f(t)$ interms of unit step function and inturn we can find its Laplace transform by using properties (i) and (ii). The following two results : (iii) and (iv) which can be easily verified will be highly useful.*

(iii) If $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a \end{cases}$

Then $f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-a)$

Proof R.H.S $= f_1(t) + [f_2(t) - f_1(t)]\begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$

$= f_1(t) + \begin{cases} 0, & t \le a \\ f_2(t) - f_1(t), & t > a \end{cases} = \begin{cases} f_1(t) + 0, & t \le a \\ f_1(t) + f_2(t) - f_1(t), & t > a \end{cases}$

ie., R.H.S $= \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a \end{cases} = f(t) = $ L.H.S *(iv)* If $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & a < t \le b \\ f_3(t), & t > b \end{cases}$

Then $f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-a) + [f_3(t) - f_2(t)]\,u(t-b)$

Proof : $\text{R.H.S} = f_1(t) + [f_2(t) - f_1(t)] \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases} + [f_3(t) - f_2(t)] \begin{cases} 0, & t \le b \\ 1, & t > b \end{cases}$

$$\text{R.H.S.} = f_1(t) + \begin{cases} 0, & t \le a \\ f_2(t) - f_1(t), & t > a \end{cases} + \begin{cases} 0, & t \le b \\ f_3(t) - f_2(t), & t > b \end{cases}$$

$$= \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a \end{cases} + \begin{cases} 0, & t \le b \\ f_3(t) - f_2(t), & t > b \end{cases}$$

$$\text{R.H.S.} = \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a,\ t \le b \\ f_3(t), & t > b \end{cases} = \begin{cases} f_1(t), & t \le a \\ f_2(t), & a < t \le b \\ f_3(t), & t > b \end{cases} = f(t) = \text{L.H.S}$$

WORKED PROBLEMS

Working procedure for problems

Type-1 : *To find* $L[F(t)\,u(t-a)]$ *where* $F(t)$ *is a polynomial in t.*

- Let $F(t) = f(t-a)$ which implies that $F(t+a) = f(t)$
- Replace t by $t+a$ to obtain $f(t)$
- Find $L[f(t)] = \bar{f}(s)$
- $L[F(t)\,u(t-a)] = L[f(t-a)\,u(t-a)] = e^{-as}\,\bar{f}(s)$ by property (ii)

Type-2 : ***Given*** $f(t)$ ***as a discontinuous function, to find*** $L[f(t)]$ ***by expressing*** $f(t)$ ***in terms of unit step function***

- We express $f(t)$ in terms of unit step function by directly making use of the result (iii) or (iv) as the case may be.
- We find $L[f(t)]$ as in Type-1

9

Find the Laplace transform of the following functions

63. $[e^{t-1}+\sin(t-1)]\,u(t-1)$

64. $\sin t\,u(t-\pi)$

65. $(3t^2+4t+5)\,u(t-3)$

66. $(1-e^{2t})\,u(t+1)$

67. $(t^3+t^2+t+1)\,u(t+1)$

68. $(t^2-6t+9)\,e^{-(t-3)}\,u(t-3)$

63. Let $f(t-1) = e^{t-1}+\sin(t-1)$

$\Rightarrow \quad f(t) = e^t+\sin t \quad \therefore \quad \bar{f}(s) = \dfrac{1}{s-1}+\dfrac{1}{s^2+1}$

We have $L[f(t-1)\,u(t-1)] = e^{-s}\,\bar{f}(s)$; $(a = 1)$

Thus $\quad L[e^{t-1}+\sin(t-1)]\,u(t-1) = e^{-s}\left[\dfrac{1}{s-1}+\dfrac{1}{s^2+1}\right]$

---

64. Let $f(t-\pi) = \sin t$

$\Rightarrow \quad f(t) = \sin(t+\pi) = -\sin t \quad \therefore \quad \bar{f}(s) = \dfrac{-1}{s^2+1}$

We have, $L[f(t-\pi)\,u(t-\pi)] = e^{-\pi s}\,\bar{f}(s)$; $(a = \pi)$

Thus $\quad L[\sin t\,u(t-\pi)] = -e^{-\pi s}/s^2+1$

---

65. Let $f(t-3) = 3t^2+4t+5$

$\Rightarrow \quad f(t) = 3(t+3)^2+4(t+3)+5 = 3t^2+22t+44$

$\therefore \quad \bar{f}(s) = 3\cdot\dfrac{2!}{s^3}+22\cdot\dfrac{1!}{s^2}+\dfrac{44}{s} = \dfrac{6}{s^3}+\dfrac{22}{s^2}+\dfrac{44}{s}$

We have, $L[f(t-3)\,u(t-3)] = e^{-3s}\,\bar{f}(s)$; $(a = 3)$

Thus $\quad L[(3t^2+4t+5)\,u(t-3)] = e^{-3s}\left[6/s^3+22/s^2+44/s\right]$

---

66. Let $f(t+1) = 1-e^{2t}$

$\Rightarrow \quad f(t) = 1-e^{2(t-1)}$ by replacing t by $(t-1)$

ie., $\quad f(t) = 1-e^{-2}\cdot e^{2t} \quad \therefore \quad \bar{f}(s) = \dfrac{1}{s}-e^{-2}\cdot\dfrac{1}{s-2}$

We have, $L[f(t+1)\,u(t+1)] = e^{s}\,\bar{f}(s)$; $(a = -1)$

Thus $\quad L[(1-e^{2t})\,u(t+1)] = e^{s}\left[\dfrac{1}{s}-\dfrac{1}{e^2(s-2)}\right] = \dfrac{e^s}{s}-\dfrac{e^{(s-2)}}{s-2}$

---

67. Let $f(t+1) = t^3 + t^2 + t + 1$

$\Rightarrow \quad f(t) = (t-1)^3 + (t-1)^2 + (t-1) + 1$

$= (t^3 - 3t^2 + 3t - 1) + (t^2 - 2t + 1) + (t - 1) + 1$

ie., $f(t) = t^3 - 2t^2 + 2t \quad \therefore \ \bar{f}(s) = \frac{3!}{s^4} - 2 \cdot \frac{2!}{s^3} + 2 \cdot \frac{1!}{s^2}$

We have, $L[f(t+1)u(t+1)] = e^s \bar{f}(s)$; $(a = -1)$

Thus $L[(t^3 + t^2 + t + 1)\ u(t+1)] = e^s \left(\frac{6}{s^4} - \frac{4}{s^3} + \frac{2}{s^2} \right)$

---

68. Let $f(t-3) = (t^2 - 6t + 9)e^{-(t-3)} = (t-3)^2 e^{-(t-3)}$

$\Rightarrow \quad f(t) = t^2 e^{-t} \quad \therefore \quad \bar{f}(s) = \frac{2!}{(s+1)^3} = \frac{2}{(s+1)^3}$

We have, $L[f(t-3)u(t-3)] = e^{-3s}\bar{f}(s)$; $(a = 3)$

Thus $L[(t^2 - 6t + 9)e^{-(t-3)}u(t-3)] = e^{-3s} \cdot \frac{2}{(s+1)^3} = \frac{2e^{-3s}}{(s+1)^3}$

Express the following functions interms of Heaviside unit step function and hence find their Laplace transform.

69. $f(t) = \begin{cases} t, & 0 < t < 4 \\ 5, & t > 4 \end{cases}$

70. $f(t) = \begin{cases} \sin 2t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

71. $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$

72. $f(t) = \begin{cases} \sin t, & 0 < t \le \pi/2 \\ \cos t, & t > \pi/2 \end{cases}$

73. $f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ t^2, & t > 2 \end{cases}$

74. $f(t) = \begin{cases} \cos t, & 0 < t \le \pi \\ 1, & \pi < t \le 2\pi \\ \sin t, & t > 2\pi \end{cases}$

75. $f(t) = \begin{cases} e^{2t}, & 0 < t < 1 \\ 2, & t > 1 \end{cases}$

76. $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > 2\pi \end{cases}$

69. $f(t) = \begin{cases} t, & 0 < t < 4 \\ 5, & t > 4 \end{cases}$

$f(t) = t + (5 - t)\, u(t - 4)$ by a property.

$$L[f(t)] = L(t) + L[(5-t)u(t-4)] \qquad \ldots (1)$$

We have $L(t) = 1/s^2$

Let $F(t-4) = (5-t) \quad \therefore \; F(t) = 5 - (t+4) = 1 - t$

Hence $\overline{F}(s) = L[F(t)] = \dfrac{1}{s} - \dfrac{1}{s^2}$

But $L[F(t-4)u(t-4)] = e^{-4s}\overline{F}(s)$

ie., $L[(5-t)u(t-4)] = e^{-4s}\left(1/s - 1/s^2\right)$

Thus (1) becomes,

$$L[f(t)] = \frac{1}{s^2} + e^{-4s}\left(\frac{1}{s} - \frac{1}{s^2}\right) = \frac{1}{s}e^{-4s} + \frac{1}{s^2}(1 - e^{-4s})$$

Remark : *Refer Problem-1, the problem has been worked by the basic definition.*

---

70. $f(t) = \begin{cases} \sin 2t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

$f(t) = \sin 2t + (0 - \sin 2t)\, u(t - \pi)$ by a property.

$$L[f(t)] = L(\sin 2t) - L[\sin 2t\, u(t-\pi)] \qquad \ldots (1)$$

Let us find $L[\sin 2t\, u(t-\pi)]$

Taking $F(t-\pi) = \sin 2t$, $F(t) = \sin 2(t+\pi) = \sin(2\pi + 2t)$

ie., $F(t) = \sin 2t \quad \therefore \; \overline{F}(s) = 2/s^2 + 4$

But $L[F(t-\pi)u(t-\pi)] = e^{-\pi s}\overline{F}(s) = 2e^{-\pi s}/s^2 + 4$

Thus (1) becomes,

$$L[f(t)] = \frac{2}{s^2+4} - \frac{2e^{-\pi s}}{s^2+4} = \frac{2(1 - e^{-\pi s})}{s^2+4}$$

Remark : *Refer Problem-2, the problem has been worked directly.*

---

71. $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$

$f(t) = \cos t + (\sin t - \cos t)\, u(t-\pi)$ by a property.

$L[f(t)] = L(\cos t) + L[(\sin t - \cos t)\, u(t-\pi)]$ ...(1)

Now let $F(t-\pi) = \sin t - \cos t$

$\Rightarrow \quad F(t) = \sin(t+\pi) - \cos(t+\pi) = -\sin t + \cos t$

$\therefore \quad \bar{F}(s) = \dfrac{-1}{s^2+1} + \dfrac{s}{s^2+1} = \dfrac{s-1}{s^2+1}$

But $\quad L[F(t-\pi)\, u(t-\pi)] = e^{-\pi s}\bar{F}(s) = \dfrac{e^{-\pi s}(s-1)}{s^2+1}$

Thus (1) becomes

$$L[f(t)] = \frac{s}{s^2+1} + \frac{e^{-\pi s}(s-1)}{s^2+1} = \frac{s + e^{-\pi s}(s-1)}{s^2+1}$$

---

72. $f(t) = \begin{cases} \sin t, & 0 < t \le \pi/2 \\ \cos t, & t > \pi/2 \end{cases}$

$f(t) = \sin t + (\cos t - \sin t)\, u(t-\pi/2)$ by a property.

$L[f(t)] = L(\sin t) + L[(\cos t - \sin t)\, u(t-\pi/2)]$ ...(1)

Now, let $\quad F(t-\pi/2) = \cos t - \sin t$

$\Rightarrow \quad F(t) = \cos(t+\pi/2) - \sin(t+\pi/2) = -\sin t - \cos t$

$\therefore \quad \bar{F}(s) = \dfrac{-1}{s^2+1} - \dfrac{s}{s^2+1} = \dfrac{-(s+1)}{(s^2+1)}$

But $\quad L[F(t-\pi/2)\, u(t-\pi/2)] = e^{-\pi s/2}\,\bar{F}(s) = \dfrac{-e^{-\pi s/2}(s+1)}{s^2+1}$

Thus (1) becomes

$$L[f(t)] = \frac{1}{s^2+1} - \frac{e^{-\pi s/2}(s+1)}{(s^2+1)} = \frac{1 - e^{-\pi s/2}(s+1)}{s^2+1}$$

---

73. $f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ t^2, & t > 2 \end{cases}$

$f(t) = 1 + (t-1)\,u(t-1) + (t^2 - t)\,u(t-2)$ by a property.

$$L[f(t)] = L(1) + L[(t-1)\,u(t-1)] + L[(t^2-t)\,u(t-2)] \quad \ldots (1)$$

Let $\quad F(t-1) = (t-1) \;;\; G(t-2) = t^2 - t$

$\Rightarrow \quad F(t) = t \;;\; G(t) = (t+2)^2 - (t+2) = t^2 + 3t + 2$

$\therefore \quad \overline{F}(s) = \dfrac{1}{s^2} \;;\; \overline{G}(s) = \dfrac{2}{s^3} + \dfrac{3}{s^2} + \dfrac{2}{s}$

$L[F(t-1)\,u(t-1)] = e^{-s}\,\overline{F}(s)$ and $L[G(t-2)\,u(t-2)] = e^{-2s}\,\overline{G}(s)$

ie., $\quad L[(t-1)\,u(t-1)] = \dfrac{e^{-s}}{s^2}$ and $L[(t^2-t)\,u(t-2)] = e^{-2s}\left(\dfrac{2}{s^3} + \dfrac{3}{s^2} + \dfrac{2}{s}\right)$

Thus (1) becomes,

$$L[f(t)] = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)$$

---

74. $f(t) = \begin{cases} \cos t, & 0 < t \le \pi \\ 1, & \pi < t \le 2\pi \\ \sin t, & t > 2\pi \end{cases}$

$f(t) = \cos t + (1 - \cos t)\,u(t-\pi) + (\sin t - 1)\,u(t - 2\pi)$

$$L[f(t)] = L(\cos t) + L[(1-\cos t)\,u(t-\pi)] + L[(\sin t - 1)\,u(t-2\pi)] \quad \ldots (1)$$

Let $\quad F(t-\pi) = 1 - \cos t \;;\; G(t-2\pi) = \sin t - 1$

$\Rightarrow \quad F(t) = 1 - \cos(t+\pi) \;;\; G(t) = \sin(t+2\pi) - 1$

ie., $\quad F(t) = 1 + \cos t \;;\; G(t) = \sin t - 1$

$\therefore \quad \overline{F}(s) = \dfrac{1}{s} + \dfrac{s}{s^2+1} \;;\; \overline{G}(s) = \dfrac{1}{s^2+1} - \dfrac{1}{s}$

$L[F(t-\pi)\,u(t-\pi)] = e^{-\pi s}\,\overline{F}(s)$ and $L[G(t-2\pi)\,u(t-2\pi)] = e^{-2\pi s}\,\overline{G}(s)$

ie., $\quad L[(1-\cos t)\,u(t-\pi)] = e^{-\pi s}\left(\dfrac{1}{s} + \dfrac{s}{s^2+1}\right)$ and

$L[(\sin t - 1)\,u(t-2\pi)] = e^{-2\pi s}\left(\dfrac{1}{s^2+1} - \dfrac{1}{s}\right)$

Thus (1) becomes,

$$L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s}\right)$$

---

75. $f(t) = \begin{cases} e^{2t}, & 0<t<1 \\ 2, & t>1 \end{cases}$

$$f(t) = e^{2t} + (2-e^{2t})\,u(t-1)$$

$$L[f(t)] = L(e^{2t}) + L[(2-e^{2t})\,u(t-1)] \quad \ldots (1)$$

Now, let $F(t-1) = 2-e^{2t} \Rightarrow F(t) = 2-e^{2(t+1)} = 2-e^2 \cdot e^{2t}$

$\therefore \quad \overline{F}(s) = \dfrac{2}{s} - e^2 \cdot \dfrac{1}{s-2}$

But $\quad L[F(t-1)\,u(t-1)] = e^{-s}\overline{F}(s)$

ie., $\quad L[(2-e^{2t})\,u(t-1)] = e^{-s}\left(\dfrac{2}{s} - \dfrac{e^2}{s-2}\right)$

Thus (1) becomes

$$L[f(t)] = \frac{1}{s-2} + e^{-s}\left(\frac{2}{s} - \frac{e^2}{s-2}\right)$$

---

76. $f(t) = \begin{cases} \cos t, & 0<t<\pi \\ \cos 2t, & \pi<t<2\pi \\ \cos 3t, & t>2\pi \end{cases}$

$f(t) = \cos t + (\cos 2t - \cos t)\,u(t-\pi) + (\cos 3t - \cos 2t)\,u(t-2\pi)$

$L[f(t)] = L(\cos t) + L[(\cos 2t - \cos t)\,u(t-\pi)]$

$$+ L[(\cos 3t - \cos 2t)\,u(t-2\pi)] \quad \ldots (1)$$

Let $\quad F(t-\pi) = \cos 2t - \cos t \quad ; \quad G(t-2\pi) = \cos 3t - \cos 2t$

$\Rightarrow \quad F(t) = \cos 2(t+\pi) - \cos(t+\pi)$ and $G(t) = \cos 3(t+2\pi) - \cos 2(t+2\pi)$

ie., $\quad F(t) = \cos 2t + \cos t \quad ; \quad G(t) = \cos 3t - \cos 2t$

$\therefore \quad \overline{F}(s) = \dfrac{s}{s^2+4} + \dfrac{s}{s^2+1} \quad ; \quad \overline{G}(s) = \dfrac{s}{s^2+9} - \dfrac{s}{s^2+4}$

But $\quad L[F(t-\pi)\,u(t-\pi)] = e^{-\pi s}\overline{F}(s)$ and

$\quad L[G(t-2\pi)\,u(t-2\pi)] = e^{-2\pi s}\,\overline{G}(s)$

ie., $$L[(\cos 2t - \cos t)\,u(t-\pi)] = e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right)$$

and $$L[(\cos 3t - \cos 2t)\,u(t-2\pi)] = e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

Hence (1) becomes

$$L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

Thus $$\boldsymbol{L[f(t)] = \frac{s}{s^2+1} + s\,e^{-\pi s}\left(\frac{1}{s^2+4} + \frac{1}{s^2+1}\right) - \frac{5\,s\,e^{-2\pi s}}{(s^2+4)(s^2+9)}}$$

ADDITIONAL PROBLEMS

77. *Find* (i) $L[e^{-t}\,u(t-2)]$ (ii) $L[t^2\,u(t-3)]$

>> (i) Let $F(t-2) = e^{-t}$

$\Rightarrow$ $F(t) = e^{-(t+2)} = e^{-2}\cdot e^{-t}$

$\therefore$ $L[F(t)] = \overline{F}(s) = e^{-2}\cdot\dfrac{1}{s+1} = \dfrac{e^{-2}}{s+1}$

We know that $L[F(t-2)\,u(t-2)] = e^{-2s}\,\overline{F}(s)$

Thus $$\boldsymbol{L[e^{-t}\,u(t-2)] = e^{-2s}\cdot\frac{e^{-2}}{s+1} = \frac{e^{-2(s+1)}}{s+1}}$$

(ii) Let $G(t-3) = t^2$

$\Rightarrow$ $G(t) = (t+3)^2 = t^2+6t+9$

$\therefore$ $L[G(t)] = \overline{G}(s) = \dfrac{2!}{s^3} + 6\cdot\dfrac{1!}{s^2} + \dfrac{9}{s} = \dfrac{2}{s^3} + \dfrac{6}{s^2} + \dfrac{9}{s}$

We know that $L[G(t-3)\,u(t-3)] = e^{-3s}\,\overline{G}(s)$

Thus $$\boldsymbol{L[t^2\,u(t-3)] = e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)}$$

---

78. *Express the following function in terms of Heaviside unit step function and hence find its Laplace transform where*

$$f(t) = \begin{cases} t^2, & 0 < t \le 2 \\ 4t, & t > 2 \end{cases}$$

>> $f(t) = t^2 + (4t - t^2)\, u(t-2)$, by a property .

$$L[f(t)] = L(t^2) + L[(4t - t^2)\, u(t-2)] \qquad \ldots (1)$$

We shall find $L[(4t - t^2)\, u(t-2)]$

Taking $F(t-2) = 4t - t^2$ we have,

$$F(t) = 4(t+2) - (t+2)^2$$

i.e., $F(t) = 4 - t^2$

Hence $\overline{F}(s) = L[F(t)] = \frac{4}{s} - \frac{2}{s^3}$

But $L[F(t-2)\, u(t-2)] = e^{-2s}\, \overline{F}(s)$, by a property.

$$\therefore \quad L[(4t - t^2)\, u(t-2)] = e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right) \qquad \ldots (2)$$

Thus by using (2) and $L(t^2) = 2/s^3$ in (1) we get

$$\boldsymbol{L[f(t)] = \frac{2}{s^3} + e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right)}$$

---

79. *Express the function* $f(t) = \begin{cases} \pi - t, & 0 < t \le \pi \\ \sin t, & t > \pi \end{cases}$ *interms of unit step function and hence find its Laplace transform*

>> $f(t) = (\pi - t) + [\sin t - (\pi - t)]\, u(t - \pi)$ by a standard property.

i.e., $f(t) = (\pi - t) + [\sin t - \pi + t]\, u(t - \pi)$

$$L[f(t)] = L(\pi - t) + L\{[\sin t - \pi + t]\, u(t - \pi)\} \qquad \ldots (1)$$

Taking $F(t - \pi) = \sin t - \pi + t$, we have

$$F(t) = \sin(t + \pi) - \pi + (t + \pi)$$

i.e., $F(t) = -\sin t + t$

$$\therefore \quad \overline{F}(s) = L[F(t)] = \frac{-1}{s^2+1} + \frac{1}{s^2}$$

Also $L[F(t-\pi)\, u(t-\pi)] = e^{-\pi s}\, \overline{F}(s)$

$$\therefore \quad L\left[(\sin t-\pi+t)\,u(t-\pi)\right] = e^{-\pi s}\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right) \qquad \ldots(2)$$

Thus by using (2) in (1), with $L(\pi-t) = \pi/s - 1/s^2$ we get,

$$Lf(t)] = \left(\frac{\pi}{s}-\frac{1}{s^2}\right) + e^{-\pi s}\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)$$

---

80. *Define Heaviside unit step function. Using unit step function find the Laplace transform of*

$$f(t) = \begin{cases} \sin t & 0 \le t < \pi \\ \sin 2t & \pi \le t < 2\pi \\ \sin 3t & t \ge 2\pi \end{cases}$$

The given $f(t)$ can be written in the following form by a standard property.

$$f(t) = \sin t + [\sin 2t - \sin t]\,u(t-\pi) + [\sin 3t - \sin 2t]\,u(t-2\pi)$$

Now $L[f(t)] = L(\sin t) + L\{[\sin 2t - \sin t]\ u(t-\pi)\}$

$$+ L\{[\sin 3t - \sin 2t]\,u(t-2\pi)\} \qquad \ldots(1)$$

Consider $L\{[\sin 2t - \sin t]\,u(t-\pi)\}$

Let $F(t-\pi) = \sin 2t - \sin t$

$\Rightarrow$ $F(t) = \sin 2(t+\pi) - \sin(t+\pi)$

ie., $F(t) = \sin(2\pi + 2t) - \sin(\pi + t)$

or $F(t) = \sin 2t + \sin t$

$$\therefore \quad \overline{F}(s) = L[F(t)] = \frac{2}{s^2+4} + \frac{1}{s^2+1}$$

But $L[F(t-\pi)\,u(t-\pi)] = e^{-\pi s}\,\overline{F}(s)$

$$ie., \quad L\{[\sin 2t - \sin t]\,u(t-\pi)]\} = e^{-\pi s}\left[\frac{2}{s^2+4} + \frac{1}{s^2+1}\right] \qquad \ldots(2)$$

Also let $G(t-2\pi) = \sin 3t - \sin 2t$

$\Rightarrow$ $G(t) = \sin 3(t+2\pi) - \sin 2(t+2\pi)$

ie., $G(t) = \sin 3t - \sin 2t$

$$\therefore \quad \overline{G}(s) = \frac{3}{s^2+9} - \frac{2}{s^2+4}$$

But $L[G(t-2\pi)\,u(t-2\pi)] = e^{-2\pi s}\,\overline{G}(s)$

ie., $$L\{[\sin 3t - \sin 2t]\, u(t-2\pi)\} = e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right] \quad \ldots (3)$$

Thus (1) as a result of (2) and (3) becomes

$$L[f(t)] = \frac{1}{s^2+1} + e^{-\pi s}\left[\frac{2}{s^2+4} + \frac{1}{s^2+1}\right] + e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]$$

7.8 Unit impulse function

The *unit impulse function* or the *Dirac delta function* $\delta(t-a)$ is defined as follows.

$$\delta(t-a) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t-a) \; ; \quad a \geq 0$$

where $$\delta_\varepsilon(t-a) = \begin{cases} \dfrac{1}{\varepsilon} & \text{if } a \leq t \leq a+\varepsilon \\ 0, & \text{otherwise} \end{cases}$$

7.81 Laplace transform of the unit impulse function

We shall first find the Laplace transform of $\delta_\varepsilon(t-a)$

$$L[\delta_\varepsilon(t-a)] = \int_0^\infty e^{-st}\, \delta_\varepsilon(t-a)\, dt$$

$$= \int_0^a e^{-st}\, \delta_\varepsilon(t-a)\, dt + \int_a^{a+\varepsilon} e^{-st}\, \delta_\varepsilon(t-a)\, dt + \int_{a+\varepsilon}^\infty e^{-st}\, \delta_\varepsilon(t-a)\, dt$$

ie., $$L[\delta_\varepsilon(t-a)] = \int_a^{a+\varepsilon} e^{-st} \cdot \frac{1}{\varepsilon}\, dt, \text{ by using the definition.}$$

$$= \frac{1}{\varepsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\varepsilon} = \frac{-1}{\varepsilon s}\left\{e^{-s(a+\varepsilon)} - e^{-as}\right\}$$

$\therefore$ $$L[\delta_\varepsilon(t-a)] = e^{-as}\left\{\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right\} \quad \ldots (1)$$

But $$L[\delta(t-a)] = L\left\{\lim_{\varepsilon \to 0} \delta_\varepsilon(t-a)\right\} = \lim_{\varepsilon \to 0} L[\delta_\varepsilon(t-a)]$$

Hence taking limit on both sides of (1) as $\varepsilon \to 0$ we have,

$$\lim_{\varepsilon \to 0} L[\delta_\varepsilon(t-a)] = e^{-as} \lim_{\varepsilon \to 0} \left\{ \frac{1-e^{-\varepsilon s}}{\varepsilon s} \right\} = e^{-as}\, k \qquad \ldots (2)$$

where $k = \lim\limits_{\varepsilon \to 0} \dfrac{1-e^{-\varepsilon s}}{\varepsilon s}$. This being in the indeterminate form $\dfrac{0}{0}$, we employ L'Hospital's rule to evaluate the limit.

Hence $k = \lim\limits_{\varepsilon \to 0} \dfrac{0-(-s)e^{-\varepsilon s}}{s} = \lim\limits_{\varepsilon \to 0} e^{-\varepsilon s} = 1$

Using $k = 1$ in (2) we have,

$$\lim_{\varepsilon \to 0} L[\delta_\varepsilon(t-a)] = e^{-as}$$

Thus $\mathbf{L[\delta(t-a)] = e^{-as}}$

Note : In particular if $a = 0$, $L[\delta(t)] = 1$

WORKED PROBLEMS

81. *Find* $L[2\delta(t-1)+3\delta(t-2)+4\delta(t+3)]$

>> We have $2L[\delta(t-1)+3L[\delta(t-2)]+4L[\delta(t+3)]$

$$= \mathbf{2e^{-s}+3e^{-2s}+4e^{3s}} \text{ since } L[\delta(t-a)] = e^{-as}$$

---

82. *Find* $L[\cos h\, 3t\, \delta(t-2)]$

>> $$\cos h\, 3t\, \delta(t-2) = \frac{1}{2}\left\{e^{3t}+e^{-3t}\right\}\delta(t-2)$$

$$L[\cosh 3t\, \delta(t-2)] = \frac{1}{2}\left\{L[e^{3t}\delta(t-2)] + L[e^{-3t}\delta(t-2)]\right\}$$

$$= \frac{1}{2}\left\{L[\delta(t-2)]_{s\to s-3} + L[\delta(t-2)]_{s\to s+3}\right\}$$

$$= \frac{1}{2}\left\{(e^{-2s})_{s\to s-3} + (e^{-2s})_{s\to s+3}\right\}$$

$$= \frac{1}{2}\left\{e^{-2(s-3)} + e^{-2(s+3)}\right\}$$

$$\therefore \quad L[\cosh 3t\, \delta(t-2)] = \frac{e^{-2s}}{2}\left\{e^{6}+e^{-6}\right\}$$

Thus $\mathbf{L[\cosh 3t\, \delta(t-2)] = \cosh 6\, e^{-2s}}$

---

83. *Find* $L[t^4\delta(t-3)]$

>> $L[\delta(t-3)] = e^{-3s}$

$\therefore$ $L[t^4\delta(t-3)] = (-1)^4 \dfrac{d^4}{ds^4}[e^{-3s}] = (-3)^4 e^{-3s}$

Thus **$L[t^4\delta(t-3)] = 81\,e^{-3s}$**

---

84. *Find* $L[(t-1)^2\delta(t-a)]$

>> $(t-1)^2\delta(t-a) = (t^2-2t+1)\delta(t-a)$

$$L[(t-1)^2\delta(t-a)] = L[t^2\delta(t-a)] - 2L[t\delta(t-a)] + L[\delta(t-a)]$$

$$= (-1)^2\frac{d^2}{ds^2}(e^{-as}) - 2(-1)^1\frac{d}{ds}(e^{-as}) + e^{-as}$$

$$= a^2e^{-as} - 2a\,e^{-as} + e^{-as} = (a^2-2a+1)\,e^{-as}$$

Thus **$L[(t-1)^2\delta(t-a)] = (a-1)^2 e^{-as}$**

---

85. *Find* $L\left[\dfrac{2\delta(t-3)+3\delta(t-2)}{t}\right]$

$L[2\delta(t-3)+3\delta(t-2)] = 2e^{-3s}+3e^{-2s}$

$\therefore$ $L\left[\dfrac{2\delta(t-3)+3\delta(t-2)}{t}\right] = \displaystyle\int_s^\infty (2e^{-3s}+3e^{-2s})\,ds$

$$= \left[2\frac{e^{-3s}}{-3} + 3\frac{e^{-2s}}{-2}\right]_s^\infty$$

$$= \frac{2}{3}e^{-3s} + \frac{3}{2}e^{-2s}$$

Thus **$L\left[\dfrac{2\delta(t-3)+3\delta(t-2)}{t}\right] = \dfrac{1}{6}(4e^{-3s}+9e^{-2s})$**

EXERCISES

Find the Laplace transform of the following functions

I 1. $f(t) = \begin{cases} 2, & 0<t<3 \\ t, & t>3 \end{cases}$ 2. $f(t) = \begin{cases} t, & 0<t<1 \\ t^2, & t>1 \end{cases}$

II 3. $\sin 3t \sin 2t \sin t$ 4. $\cos(2t+3) + \cos 7t \cos 3t$

5. $4\sin^2 t \cos t$ 6. $t\sqrt{t} + 4t^3 + 3^t$

III 7. $e^{-t}\cos^2 3t$ 8. $e^{-3t}\sin^3 2t$ 9. $\cosh 2t \cos 2t$

IV 10. $t\sin^2 t$ 11. $t \sin 3t \cos t$ 12. $(1 + t e^t)^3$

V 13. $\dfrac{\cosh at - \cos bt}{t}$ 14. $\dfrac{1-\cos at}{t}$ 15. $\dfrac{2\sin 3t \cos 5t}{t}$

VI 16. $\int_0^t \frac{\sin at}{t}\,dt$ 17. $\int_0^t e^{-t}\cos t\,dt$ 18. $\int_0^t t e^{-t}\sin 4t\,dt$

VII *Evaluate the following integrals using Laplace transforms*

19. $\int_0^\infty e^{-3t}\, t \sin t\,dt$ 20. $\int_0^\infty t^2 e^t \cos t\,dt$

21. $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\,dt$ 22. $\int_0^\infty \frac{\sin t}{t}\,dt$

VIII *Find the Laplace transform of the following functions*

23. $f(t) = \begin{cases} a, & 0 \le t \le a \\ -a, & a<t<2a \end{cases}$ 24. $f(t) = \begin{cases} t, & 0\le t \le \pi \\ 2\pi - t, & \pi \le t \le 2\pi \end{cases}$

where $f(t+2a) = f(t)$ where $f(t+2\pi) = f(t)$

25. $f(t) = E \sin \omega t$ in $0 < t < \pi/\omega$

IX *Find the Laplace transform of the following functions*

26. $(t^2 + 2t - 1)\,u(t-3)$ 27. $(\sin t + \cos t)\,u(t - \pi/2)$

28. $e^{-t}\,u(t-2)$

X *Express the following functions interms of Heaviside unit step function and hence find its Laplace transform*

29. $f(t) = \begin{cases} t^2, & 1 < t \le 2 \\ 4t, & t > 2 \end{cases}$

30. $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ \sin 2t, & \pi < t < 2\pi \\ \sin 3t, & t > 2\pi \end{cases}$

XI *Find the Laplace transform of the following functions*

31. $\cosh t\, \delta(t-a)$

32. $\sinh 3t\, \delta(t-2)$

33. $t^n\, \delta(t-3)$

34. $(t+1)^2\, \delta(t-2)$

35. $[\,2\,\delta(t-1) + 6\,\delta(t-2)\,]/t$

ANSWERS

I

1. $\dfrac{2+e^{-3s}}{s} + \dfrac{e^{-3s}}{s^2}$

2. $\dfrac{1+e^{-s}}{s^2} + \dfrac{2\,e^{-s}}{s^3}$

II

3. $\dfrac{1}{2}\left[\dfrac{1}{s^2+4} - \dfrac{3}{s^2+36} + \dfrac{2}{s^2+16}\right]$

4. $\cos 3 \cdot \dfrac{s}{s^2+4} - \sin 3 \cdot \dfrac{2}{s^2+4} + \dfrac{s}{2}\left(\dfrac{1}{s^2+100} + \dfrac{1}{s^2+16}\right)$

5. $\dfrac{8s}{(s^2+1)(s^2+9)}$

6. $\dfrac{3\sqrt{\pi}}{4\,s^{5/2}} + \dfrac{24}{s^4} + \dfrac{1}{s-\log 3}$

III

7. $\dfrac{s^2+2s+19}{(s+1)(s^2+2s+37)}$

8. $\dfrac{48}{(s^2+6s+13)(s^2+6s+45)}$

9. $\dfrac{s^3}{s^4+64}$

IV

10. $\dfrac{2(3s^2+4)}{s^2(s^2+4)^2}$

11. $\dfrac{6s(s^4+16s^2+96)}{(s^2+16)^2(s^2+4)^2}$

12. $\dfrac{1}{s} + \dfrac{3}{(s-1)^2} + \dfrac{6}{(s-2)^3} + \dfrac{6}{(s-3)^4}$

V

13. $\log \sqrt{(s^2+b^2)/(s^2-a^2)}$

14. $\log\left(\sqrt{s^2+a^2/s^2}\right)$

15. $\tan^{-1}(s/2) - \tan^{-1}(s/8)$

VI 16. $\frac{1}{s}\cot^{-1}(s/a)$ 17. $\frac{s+1}{s(s^2+2s+2)}$

18. $\frac{8(s+1)}{s(s^2+2s+17)}$

VII 19. 3/50 20. 1 21. log 3 22. $\pi/2$

VIII 23. $a/s \cdot \tanh(as/2)$ 24. $1/s^2 \cdot \tanh(\pi s/2)$

25. $\frac{E\omega}{s^2+\omega^2}\coth(\pi s/2\omega)$

IX 26. $e^{-3s}\left(\frac{2}{s^3}+\frac{8}{s^2}+\frac{14}{s}\right)$ 27. $\frac{(s-1)e^{-\pi s/2}}{s^2+1}$

28. $\frac{e^{-2s}}{e^2(s+1)}$

X 29. $\frac{2}{s^3}+\left(\frac{4}{s}-\frac{2}{s^3}\right)e^{-2s}$

30. $\frac{1}{s^2+1}+e^{-\pi s}\left(\frac{2}{s^2+4}+\frac{1}{s^2+1}\right)+e^{-2\pi s}\left(\frac{3}{s^2+9}-\frac{2}{s^2+4}\right)$

XI 31. $\cosh a \cdot e^{-as}$ 32. $\sinh 6 \cdot e^{-2s}$

33. $3^n e^{-3s}$ 34. $9e^{-2s}$

35. $2e^{-s}+3e^{-2s}$